

Coloured Neretin Groups

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February 28, 2017

Abstract

We study almost automorphism groups of trees obtained from universal groups constructed by Burger and Mozes. A special case is Neretin's group, where the universal group is the full automorphism group of the tree. We show that, as for Neretin's group, these almost automorphism groups are compactly generated, virtually simple and in many cases have no lattices.

1 Introduction

Yurii A. Neretin constructed groups which “from the point of view of the theory of representations should resemble the group of diffeomorphisms of a circle” ([Ner93]). One of them, or more precisely one class of them, for which Neretin suggested the term “group of hierarchomorphisms” (see [Ner03]), is known under the names “Neretin's group of tree spheromorphisms”, “group of almost automorphisms of a tree” or simply “Neretin's group”. This group was proven to be simple by Kapoudjian [Kap99] and subsequently attracted the interest of group theorists. For an introduction see e.g. [GL15].

Provided with a natural topology Neretin's group is totally disconnected and locally compact. Caprace and De Medts [CDM11] found a dense copy of a Higman-Thompson group inside it and concluded that Neretin's group is compactly generated. Le Boudec [LB16] proved that it is even compactly presented. This result was strengthened by Sauer and Thumann [ST15], who showed that Neretin's group admits a cellular action on a contractible cellular complex with compact open stabilizers and such that the restriction of the action on each n -skeleton is cocompact.

Bader, Caprace, Gelander and Mozes [BCGM12] proved that Neretin's group does not have any lattice. Let Λ be a locally compact group. A lattice in Λ is a discrete subgroup $\Gamma \leq \Lambda$ such that the quotient Λ/Γ has a finite Λ -invariant measure. Lattices play a tremendously important role in geometric group theory. Neretin's group was the first known example of a locally compact simple group not admitting any lattice. Other examples, which are acting on trees, were constructed by Le Boudec [LB16]. Besides being interesting by itself, being simple in combination with having no lattice

is a necessary condition to also not admit any nontrivial invariant random subgroup (IRS). So far no example of a non-discrete locally compact group without nontrivial IRS is known. It is an open question whether Neretin's group has a nontrivial IRS. A good introduction into IRS's are for example the notes of Gelander [Gel15].

In this paper we study a generalization of Neretin's group and obtain more examples of locally compact, compactly generated, simple groups without lattices.

Let \mathcal{T} be a regular tree of degree $d+1$ and denote by $\text{Aut}(\mathcal{T})$ its group of automorphisms. The group topology on $\text{Aut}(\mathcal{T})$ is generated by all vertex stabilizers, which then turn out to be not only open, but also compact. For every vertex v of \mathcal{T} we fix a bijection from the $d+1$ edges incident to v to the set $D := \{0, \dots, d\}$. Now we can describe for every element $g \in \text{Aut}(\mathcal{T})$ its local action, i.e. for every vertex its action on the $d+1$ edges incident to it, in terms of elements of $\text{Sym}(D)$. Let $F \leq \text{Sym}(D)$ be a subgroup of the symmetric group on $d+1$ letters. Burger and Mozes [BM00] constructed closed subgroups $U(F) \leq \text{Aut}(\mathcal{T})$, called *universal groups*, whose local action at every vertex is prescribed by F . Let $\partial\mathcal{T}$ be the boundary, also called the absolute or set of ends, of \mathcal{T} . It is a Cantor space. The group $\text{Aut}(\mathcal{T})$, and therefore also the group $U(F)$, acts by homeomorphisms on $\partial\mathcal{T}$.

We define the subgroup $\mathcal{N}_F < \text{Homeo}(\partial\mathcal{T})$ consisting of all those homeomorphisms which “locally look like” elements of $U(F)$. One way to make this precise would be to say that \mathcal{N}_F is the topological full group of $U(F)$ acting on $\partial\mathcal{T}$. See Section 2.3 for precise definitions. We show that there is a unique group topology on \mathcal{N}_F such that the inclusion $U(F) \hookrightarrow \mathcal{N}_F$ is continuous and open.

Definition 1.1. A subgroup $F \leq \text{Sym}(D)$ is called a *single switch group* if it has order 2 and its only nontrivial element is a transposition.

The main theorem of this paper is the following.

Theorem 1.2. *The following hold for \mathcal{N}_F .*

- *The commutator subgroup $D(\mathcal{N}_F)$ of \mathcal{N}_F is open, simple and has finite index. More precisely, the abelianization $\mathcal{N}_F/D(\mathcal{N}_F)$ is a quotient of $(\mathbb{Z}/2\mathbb{Z})^{D/F}$.*
- *The group \mathcal{N}_F is compactly generated.*
- *If F does not act freely on D and F is not a single switch group, then \mathcal{N}_F does not have any lattice. If F is a single switch group, then \mathcal{N}_F does not have any cocompact lattice.*

Consequently, if F does not act freely on D and F is not a single switch group, then $D(\mathcal{N}_F)$ is a compactly generated, non-discrete, simple group without lattices.

In the proof of the first two statements we use results by Matui about topological full groups associated to shifts of finite type to find a dense finitely generated subgroup of \mathcal{N}_F . This is the generalization of the fact that Neretin's group contains dense copies of a Higman-Thompson group. The proof of the third statement follows the approach that Bader, Caprace, Gelander and Mozes use to prove that Neretin's group does not have any lattice. We do not know if for a single switch group F the group \mathcal{N}_F has a non-cocompact lattice.

In the appendix we focus on a similar type of groups introduced by Caprace and De Medts in [CDM11]. Let $\mathcal{T}_{d,k}$ be a rooted tree where the root has valency k and all other vertices have valency $d+1$. Let $F \leq \text{Sym}(d)$ be a subgroup. The groups we consider are obtained by cutting out finite subtrees of $\mathcal{T}_{d,k}$ and rearranging the complement in a way that is locally prescribed by F .

Namely, let $\text{Aut}_F(\mathcal{T}_{d,k})$ be the group of all automorphisms of $\mathcal{T}_{d,k}$ that fix the root and whose action on the d children of v is locally prescribed by F for every vertex v of $\mathcal{T}_{d,k}$ which is not the root. This is a closed subgroup of $\text{Aut}(\mathcal{T}_{d,k})$. Let $T, T' \subset \mathcal{T}_{d,k}$ be two finite subtrees containing the root. The complement $\mathcal{T}_{d,k} \setminus T$ is a forest, the same with T' . Consider the set of all forest isomorphisms $\varphi: \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ such that for every vertex v of $\mathcal{T}_{d,k} \setminus T$ the action of φ on the children of v is locally prescribed by F . These forest isomorphisms can be viewed as homeomorphisms of $\partial\mathcal{T}_{d,k}$. They form a group denoted $\text{AAut}_F(\mathcal{T}_{d,k}) < \text{Homeo}(\partial\mathcal{T}_{d,k})$. There exists a locally compact group topology such that $\text{Aut}_F(\mathcal{T}_{d,k}) \hookrightarrow \text{AAut}_F(\mathcal{T}_{d,k})$ is an continuous and open, see [LB14] Section 4.2. With this topology $\text{AAut}_F(\mathcal{T}_{d,k})$ is discrete if and only if $F = \{id\}$.

Neretin's group is the special case where $k = d+1$ and $F = \text{Sym}(d)$. I am grateful to Adrien Le Boudec for pointing out that if $F' \leq \text{Sym}(d+1)$ is transitive and $F \leq \text{Sym}(d)$ is a point stabilizer in F' , then $\mathcal{N}_{F'} \cong \text{AAut}_F(\mathcal{T}_{d,2})$. Many results about Neretin's group were actually formulated and proven for $\text{AAut}_F(\mathcal{T}_{d,k})$, like compact presentability, by Le Boudec [LB14], or its strengthening, by Sauer and Thumann [ST15]. We prove the following theorem.

Theorem 1.3. *If $F \neq \{id\}$, then the group $\text{AAut}_F(\mathcal{T}_{d,k})$ does not admit any lattice.*

The proof goes along the lines of [BCGM12] without any additional ideas or techniques. Still, to the author's knowledge and surprise it seems to not have been observed before.

1.1 Organization of the paper

In Section 2 we set up basic notations, definitions and terminology about trees. We introduce universal groups, almost automorphism groups and Higman-Thompson groups. We also define the group \mathcal{N}_F and show the existence of a locally compact group topology on a class of almost automorphism groups including \mathcal{N}_F .

In Section 3 we investigate a certain subgroup $V_F \leq \mathcal{N}_F$ which we think of as an analog of the Higman-Thompson group. We introduce the topological full group of a groupoid associated to a one-sided irreducible shift of finite type. We construct such a shift with the property that the topological full group of the associated groupoid is isomorphic to V_F .

In Section 4 we show that V_F is dense in \mathcal{N}_F and conclude that \mathcal{N}_F is compactly generated and its commutator subgroup is open, simple and has finite index.

In Section 5 we prove the third part of Theorem 1.2.

In the appendix we prove that the group $\text{AAut}_F(\mathcal{T}_{d,k})$ does not have any lattice.

Section 5 and the appendix do not rely on Sections 3 and 4 and can be read independently.

1.2 Acknowledgements

I am grateful to Marc Burger for helpful discussions and comments on previous versions of this work, Mario Šikić for careful reading of early versions and help with computer algebra systems, David Robertson for fruitful discussions, and Nicolás Matte Bon, Adrien Le Boudec, Pierre-Emmanuel Caprace and Stephan Tornier for useful remarks about previous versions of this article. I want to acknowledge the hospitality of the Australian Winter of Disconnectedness in Creswick, Victoria and Newcastle, New South Wales, where part of this work was completed.

2 Preliminaries

2.1 Trees

In this subsection we establish notations and conventions about graphs and trees.

Serre's definition of a graph and some terminology. A *graph* consists of a set V , called the *vertex set*, a set E , called the *edge set*, a fixed point free inversion $E \rightarrow E$, $e \mapsto \bar{e}$ and two maps $o, t: E \rightarrow V$, called *origin* and *terminus*, satisfying $o(e) = t(\bar{e})$. A *geometric edge* is a pair $\{e, \bar{e}\}$. For every edge e (or geometric edge $\{e, \bar{e}\}$) the vertex $o(e)$ and e are said to be

incident to each other. The *valency* of a vertex v is the cardinality $|o^{-1}(v)|$. A vertex of valency 1 is called *leaf*. For every edge e , the vertices $o(e)$ and $t(e)$ are called *adjacent* or *neighbours*.

An *orientation* on a graph $\mathbf{g} = (V, E)$ is a choice of representant for every geometric edge, i.e. a subset $E' \subset E$ such that for every $e \in E$ exactly one of e, \bar{e} is in E' .

A *morphism* between two graphs $\mathbf{g} = (V, E)$ and $\tilde{\mathbf{g}} = (\tilde{V}, \tilde{E})$ consists of two maps $\varphi_1: V \rightarrow \tilde{V}$ and $\varphi_2: E \rightarrow \tilde{E}$ such that for all $e \in E$ holds $o(\varphi_2(e)) = \varphi_1(o(e))$. Note that this implies that a morphism is uniquely determined by φ_2 . If E' and \tilde{E}' are orientations on \mathbf{g} respectively $\tilde{\mathbf{g}}$ the morphism is called *oriented* if $\varphi_2(E') \subset \tilde{E}'$.

Paths. Let $n \in \mathbb{N}$. Define the oriented graph Path_n as follows. Its vertex set is $\{0, \dots, n\} \subset \mathbb{N}$. Its edge set is $\bigcup_{i=0}^{n-1} \{[i, i+1], \overline{[i, i+1]}\}$ and the orientation on Path_n is given by $\{[i, i+1] \mid i = 0, \dots, n-1\}$. Similarly define Path_∞ to be the graph with vertex set \mathbb{N} , edge set $\bigcup_{i=0}^{\infty} \{[i, i+1], \overline{[i, i+1]}\}$ and orientation $\{[i, i+1] \mid i \in \mathbb{N}\}$.

Let $\mathbf{g} = (V, E)$ be an (oriented) graph (and $E' \subset E$ its orientation). Let $n \in \mathbb{N}$. An *(oriented) path of length n* is an (oriented) graph morphism $\text{Path}_n \rightarrow \mathbf{g}$. An *(oriented) infinite path* is an (oriented) graph morphism $\text{Path}_\infty \rightarrow \mathbf{g}$.

We can write (oriented) paths of positive or infinite length as sequence $(e_i)_i$ in E (respectively E') such that $t(e_i) = o(e_{i+1})$. For a finite (oriented) path (e_0, \dots, e_n) we say the path is a path *from $o(e_0)$ to $t(e_n)$* and call the vertex $t(e_0, \dots, e_n) := t(e_n)$ its *endpoint*. A path of length 0 can be viewed as an element of V . It is its own endpoint.

Regular trees. For a path (e_0, \dots, e_n) as above, a pair of the form (e_i, \bar{e}_i) is called *backtracking*. A graph is called a *tree* if for every two vertices v, w there exists a path from v to w and if every path without backtracking is injective on the vertices. Paths without backtracking in a tree are called *geodesics*.

We will talk about two kinds of trees. Let $d \geq 0$ be an integer.

- A d -regular tree is a tree where every vertex has valency d .
- A d -regular rooted tree is a tree where one vertex, called the *root*, has valency d and all other vertices have valency $d + 1$.

In the whole paper $d \geq 2$ is a positive integer and $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ denotes a $(d + 1)$ -regular tree with vertex set \mathcal{V} and edge set \mathcal{E} . Endow \mathcal{T} with the usual metric such that for all $v, w \in \mathcal{V}$ the distance between v and w is the length of the geodesic from v to w . Fix a vertex $v_0 \in \mathcal{V}$ of \mathcal{T} and consider \mathcal{T} as a rooted tree with root v_0 . Now we can talk about the parent and the d children of a vertex of \mathcal{T} , namely its neighbours closer respectively more

distant to v_0 (only v_0 does not have a parent but $d + 1$ children). A path starting at v_0 is called *rooted*.

Definition 2.1. The *boundary* of \mathcal{T} is the set of all rooted infinite geodesics in \mathcal{T} . It is denoted by $\partial\mathcal{T}$.

Topology on $\partial\mathcal{T}$. For every vertex $v \in \mathcal{V}$ we denote by \mathcal{T}_v the subtree of \mathcal{T} whose vertices are all $w \in \mathcal{V}$ such that v lies on the rooted geodesic to w . It is a d -regular rooted tree with root v . Its boundary $\partial\mathcal{T}_v$ is a subset of $\partial\mathcal{T}$ in an obvious way. The set $\{\partial\mathcal{T}_v \mid v \in \mathcal{V}\}$ is a basis of the topology on $\partial\mathcal{T}$. With this topology $\partial\mathcal{T}$ is a Cantor space.

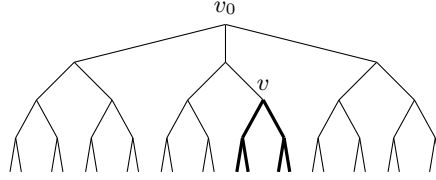


Figure 1: The thick lines indicate the subtree \mathcal{T}_v .

Automorphisms of \mathcal{T} . Denote by $\text{Aut}(\mathcal{T})$ the group of automorphisms of \mathcal{T} , that is, all graph morphisms $\mathcal{T} \rightarrow \mathcal{T}$ which are bijective on \mathcal{V} and \mathcal{E} . We define a group topology on $\text{Aut}(\mathcal{T})$ making it into a totally disconnected locally compact group. For a subset $L \subset \mathcal{V} \cup \mathcal{E}$ the *subtree spanned by L* is the unique minimal subtree of \mathcal{T} containing L . Let $G \leq \text{Aut}(\mathcal{T})$. Denote by $\text{Fix}_G(L) \leq G$ all the elements of G which fix every element of L . A neighbourhood basis of the identity in $\text{Aut}(\mathcal{T})$ consists of all subgroups of the form $\text{Fix}_{\text{Aut}(\mathcal{T})}(L)$ with $L \subset \mathcal{V} \cup \mathcal{E}$ finite. With this topology each of these basis elements is compact and open.

Notation 2.2. A subset of a topological space is *clopen* if it is closed and open.

Tits' Independence Property. Tits [Tit70], Section 4.2, defined a property for subgroups of $\text{Aut}(\mathcal{T})$ and proved a simplicity theorem for groups satisfying it. Let L be any path in \mathcal{T} . For every vertex $v \in \mathcal{V}$ denote by $\pi(v)$ the unique vertex of L which is closest to v . For such a vertex w let L_w be the subtree spanned by $\pi^{-1}(w)$. Let $G \leq \text{Aut}(\mathcal{T})$. The group $\text{Fix}_G(L)$ leaves L_w invariant. Thus, for every $g \in \text{Fix}_G(L)$ the restriction $|_{L_w}: \text{Fix}_G(L) \rightarrow \text{Aut}(L_w)$ is a well-defined homomorphism. The group G is said to have *Tits' Independence Property* if for every L as above the induced map $\text{Fix}_G(L) \rightarrow \prod_w \text{Fix}_G(L)|_{L_w}$ is an isomorphism.

Remark 2.3. If $G \leq \text{Aut}(\mathcal{T})$ is closed, Tits' Independence Property is equivalent to each of the following conditions, which for non-closed G are weaker in general:

- replacing “any path L ” by “any finite subtree L of \mathcal{T} ”;
- replacing “any path L ” by “any edge L of \mathcal{T} ”.

See for example [Ama03], page 10.

The importance of Tits' Independence Property lies in the following theorem.

Theorem 2.4 ([Tit70], Theorem 4.5). *Let $G \leq \text{Aut}(\mathcal{T})$ be a subgroup satisfying Tits' Independence Property. Assume that G neither preserves any proper subtree nor fixes any end of \mathcal{T} . Then, the subgroup*

$$G^+ := \langle \{\text{Fix}_G(e) \mid e \in \mathcal{E}\} \rangle \leq G$$

generated by all edge fixators in G is simple or trivial.

2.2 Colorings and universal groups

Definitions and statements presented here are, unless otherwise stated, due to Burger and Mozes [BM00], Section 3.2. For a more detailed introduction and proofs we refer to [GGT16], Section 4.

Legal colourings. Throughout the paper we denote $D := \{0, \dots, d\}$ and call it the *set of colours*. We fix a *legal edge colouring* of \mathcal{T} , that is a map

$$\text{col}: \mathcal{E} \rightarrow D$$

satisfying the following two properties.

- It is constant on geometric edges, i.e. for all $e \in \mathcal{E}$ holds $\text{col}(e) = \text{col}(\bar{e})$.
- For every $v \in \mathcal{V}$ the edges incident to v all have different colours, i.e. the restriction $\text{col}|_{o^{-1}(v)}: o^{-1}(v) \rightarrow D$ is a bijection.

Let $F \leq \text{Sym}(D)$ be any subgroup. Every automorphism $g \in \text{Aut}(\mathcal{T})$ induces for each vertex $v \in \mathcal{V}$ a permutation $\text{prm}_{g,v} \in \text{Sym}(D)$ defined by $\text{prm}_{g,v}(\chi) = \text{col}(g((\text{col}|_{o^{-1}(v)})^{-1}(\chi)))$.

Definition 2.5. The *universal group* associated to F is defined by

$$U(F) = \{g \in \text{Aut}(\mathcal{T}) \mid \forall v \in \mathcal{V}: \text{prm}_{g,v} \in F\}.$$

Informally speaking, it consists of all tree automorphisms whose local action is everywhere prescribed by F .

That this indeed defines a group is due to the following lemma.

Lemma 2.6 ([GGT16], Lemma 4.2.). *Let $g, h \in U(F)$ and $v \in \mathcal{V}$. Then $\text{prm}_{gh,v} = \text{prm}_{g,hv} \circ \text{prm}_{h,v}$.*

Remark 2.7. A different choice of a legal colouring will result in a universal group that is conjugate to the original one.

Remark 2.8. For every $F \leq \text{Sym}(D)$ the universal group $U(F)$ is a closed subgroup of $\text{Aut}(\mathcal{T})$ satisfying Tits' Independence Property. It is not hard to see that the group $U(F)$ is discrete if and only if the action $F \curvearrowright D$ is free, which is again equivalent to $U(F)^+ = \{1\}$.

The following lemma about extending certain tree automorphisms to “almost being in $U(F)$ ” was formulated by Le Boudec in the case of a ball around a vertex. A close look at the proof shows that it is valid for every subtree of \mathcal{T} .

Lemma 2.9 ([LB16], Lemma 3.4.). *Let T be a subtree of \mathcal{T} . Let $h \in \text{Aut}(\mathcal{T})$ be such that for every vertex v of T the permutation $\text{prm}_{h,v}$ preserves the orbits of F . Then there exists $g \in \text{Aut}(\mathcal{T})$ such that $g|_T = h|_T$ and such that for all vertices $w \in \mathcal{V}$ which are either leaves of T or not vertices of T holds $\text{prm}_{g,v} \in F$.*

Notation 2.10. We will find it convenient to also talk about coloured vertices. So, we extend the map col to $\mathcal{V} \setminus \{v_0\}$ by colouring a vertex $v \in \mathcal{V} \setminus \{v_0\}$ in the colour of the edge connecting v to its parent.

2.3 Almost automorphisms

Definition 2.11. A finite subtree $T \subset \mathcal{T}$ is called *complete* if it contains the root v_0 and if every vertex that is not a leaf has valency $d + 1$.

Notation 2.12. For a subtree $T \subset \mathcal{T}$ we will denote by $\mathcal{LT} \subset \mathcal{V}$ the set of leaves of T .

Notation 2.13. For a finite complete subtree $T \subset \mathcal{T}$ the difference $\mathcal{T} \setminus T$ will always denote the subgraph $\bigsqcup_{v \in \mathcal{LT}} \mathcal{T}_v \subset \mathcal{T}$. Hence $\mathcal{T} \setminus T$ is a forest with $|\mathcal{LT}|$ many connected components, each of which is a d -regular rooted tree.

Definition 2.14. Let T_1 and T_2 be finite complete subtrees of \mathcal{T} such that $|\mathcal{LT}_1| = |\mathcal{LT}_2|$. An *honest almost automorphism* of \mathcal{T} is a forest isomorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$.

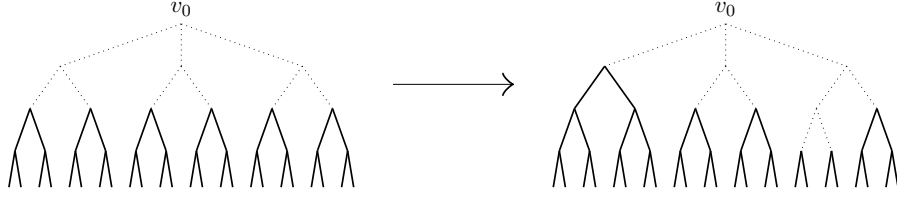


Figure 2: The trees T_1 and T_2 are indicated with dotted lines.

Almost automorphisms. We now construct an equivalence relation on the set of honest almost automorphisms. Let T_1, T_2, T'_1, T'_2 be finite complete subtrees of \mathcal{T} satisfying $\mathcal{L}T_1 = \mathcal{L}T_2$ and $\mathcal{L}T'_1 = \mathcal{L}T'_2$. Let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ and $\psi: \mathcal{T} \setminus T'_1 \rightarrow \mathcal{T} \setminus T'_2$ be honest almost automorphisms of \mathcal{T} . We say that φ and ψ are *equivalent* if there exists a finite complete subtree $T \supset T_1 \cup T'_1$ such that $\varphi|_{\mathcal{T} \setminus T} = \psi|_{\mathcal{T} \setminus T}$.

An *almost automorphism* of \mathcal{T} is the equivalence class of an honest almost automorphism under this equivalence relation. In our notation we will usually not distinguish between an honest almost automorphisms and its equivalence class, but say it explicitly whenever we need to talk about an honest almost automorphism.

Simple expansions. In proofs it will be convenient to work with generators for this equivalence relation. For finite complete subtrees $T \subset T' \subset \mathcal{T}$, we say that T' is obtained from T by a *simple expansion* if there exists a leaf v of T such that T' is spanned by T and the children of v . Note that any finite complete subtree of \mathcal{T} containing T is obtained from T by a sequence of simple expansions. If in the preceding paragraph we require that T'_1 is obtained from T_1 by a simple expansion and $T = T'_1$, the resulting relation generates the equivalence relation.

Remark 2.15. Let T_1, T_2 be finite subtrees with the same number of leaves and let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ be an honest almost automorphism. Then, for every finite complete subtree $T \subset \mathcal{T}$ containing T_1 there exists a unique finite complete subtree $T' \subset \mathcal{T}$ containing T_2 and a unique representative $\psi: \mathcal{T} \setminus T \rightarrow \mathcal{T} \setminus T'$ of φ . Explicitly $T' = \varphi(T \setminus T_1) \cup T_2$ and $\psi = \varphi|_{\mathcal{T} \setminus T}$.

The analogous statement holds for $T \supset T_2$.

Product of two almost automorphisms. Take finite complete subtrees $T_1, \dots, T_4 \subset \mathcal{T}$ such that for the number of leaves holds $|\mathcal{L}T_1| = |\mathcal{L}T_2|$ and $|\mathcal{L}T_3| = |\mathcal{L}T_4|$. Let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ and $\psi: \mathcal{T} \setminus T_3 \rightarrow \mathcal{T} \setminus T_4$ be almost automorphisms. By the previous remark we can choose a finite complete subtree $T \supset T_4 \cup T_1$ of \mathcal{T} and take representatives for ψ and φ with image respectively domain $\mathcal{T} \setminus T$. These representatives we can compose. The equivalence class of this composition is the product $\varphi \circ \psi$. With this

product the set of almost automorphisms of \mathcal{T} is a group, called *Neretin's group of tree spheromorphisms* or *the tree almost automorphism group*. We will denote it by \mathcal{N} .

The group of almost automorphisms for a subgroup of $\text{Aut}(\mathcal{T})$. Let $G \leq \text{Aut}(\mathcal{T})$ be a subgroup. We define its group $\mathcal{F}(G)$ of almost automorphisms. Let $T_1, T_2 \subset \mathcal{T}$ be finite complete subtrees satisfying $|\mathcal{L}T_1| = |\mathcal{L}T_2|$. A *G-honest almost automorphism* of \mathcal{T} is an honest almost automorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ such that for every $v \in \mathcal{L}T_1$ there exists a $g_v \in G$ with $\varphi|_{\mathcal{T}_v} = g_v|_{\mathcal{T}_v}$. The elements of $\mathcal{F}(G)$ are the equivalence classes of all *G-honest almost automorphisms*. It is not hard to see that $\mathcal{F}(G)$ is a subgroup of \mathcal{N} .

Remark 2.16. Note that Remark 2.15 remains true for *G-honest almost automorphisms*.

Remark 2.17. Let $T_1, T_2 \subset \mathcal{T}$ be finite complete subtrees with the same number of leaves. Let $G \leq \text{Aut}(\mathcal{T})$ be a subgroup. It is possible that there does not exist any *G-honest almost automorphism* $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$. For example, if $G = \text{Fix}_{\text{Aut}(\mathcal{T})}(v_0)$, then there is no *G-honest almost automorphism* as indicated in Figure 2 because every *G-honest almost automorphism* $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ needs to preserve the distance of the leaves of T_1 to v_0 .

Remark 2.18. For a subgroup $G \leq \text{Aut}(\mathcal{T})$, sometimes the intersection $\mathcal{F}(G) \cap \text{Aut}(\mathcal{T})$ is of interest. In general it is strictly larger than G and can be much different. For example, even if G is a closed subgroup of $\text{Aut}(\mathcal{T})$, this is in general not the case for $\mathcal{F}(G) \cap \text{Aut}(\mathcal{T})$.

It is easy to see that $\mathcal{F}(G) \cap \text{Aut}(\mathcal{T})$ it enjoys a weaker form of Tits' Independence Property, where L is replaced by arbitrary finite subtrees, see Remark 2.3.

Le Boudec investigated this intersection for $G = U(F)$ in [LB16] and for more general G in [LB15], Section 4. We now present one result about this intersection which will be useful later. It can be thought of as converse to Lemma 2.9.

Lemma 2.19 ([LB16], Lemma 3.3.). *Let $g \in \mathcal{F}(U(F)) \cap \text{Aut}(\mathcal{T})$. Then for every $v \in \mathcal{V}$ the permutation $\text{prm}_{g,v}$ preserves the orbits of F .*

Remark/Warning. Let $G \leq \text{Aut}(\mathcal{T})$ be a subgroup. Consider an honest almost automorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ for finite complete subtrees T_1, T_2 of \mathcal{T} such that the equivalence class of φ is an element of $\mathcal{F}(G)$. It is in general not true that φ is a *G-honest almost automorphism*.

For example, every element of $\mathcal{F}(G) \cap \text{Aut}(\mathcal{T})$ which is not an element of G is a tree automorphism and therefore has an honest almost automorphism $\mathcal{T} \setminus \emptyset \rightarrow \mathcal{T} \setminus \emptyset$ representing it. This representative is however not

a G -honest almost automorphism. Strictly speaking, to talk about honest almost automorphisms we required the trees T_1, T_2 to be complete, which implied being non-empty, but we use the empty tree here just to illustrate our warning.

The group $\mathcal{F}(G)$ as topological full group. We now give an alternative description of $\mathcal{F}(G)$ and prove that it is equivalent to the previous one.

Definition 2.20. Consider a group Λ acting on a topological space X . The *topological full group* of this action is the following subgroup of $\text{Homeo}(X)$. It consists of all those homeomorphisms $\varphi: X \rightarrow X$ such that for every $x \in X$ there exists a neighbourhood U of x and a group element $g \in \Lambda$ with $\varphi|_U = g|_U$.

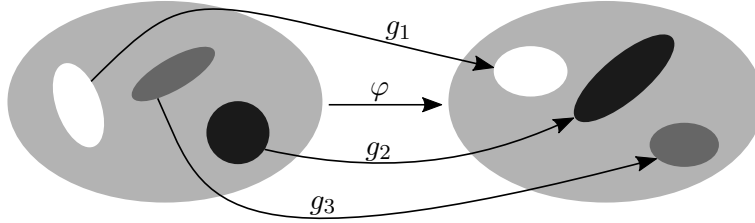


Figure 3: Illustration of the topological full group

Lemma 2.21. For a subgroup $G \leq \text{Aut}(\mathcal{T})$ the group $\mathcal{F}(G)$ is isomorphic to the topological full group of G acting on $\partial\mathcal{T}$. Consequently, it acts itself faithfully on $\partial\mathcal{T}$.

Proof. The group $\mathcal{F}(G)$ can be seen as a subgroup of $\text{Homeo}(\partial\mathcal{T})$ in the same way as G . Let T_1, T_2 be finite complete subtrees of \mathcal{T} with the same number of leaves and let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ be a G -honest almost automorphism of \mathcal{T} . Let $v \in \mathcal{L}T_1$. Then there exists a $g_v \in G$ with $\varphi|_{\mathcal{T}_v} = g_v|_{\mathcal{T}_v}$. For the action of φ on $\partial\mathcal{T}$ this means $\varphi|_{\partial\mathcal{T}_v} = g_v|_{\partial\mathcal{T}_v}$. Therefore the equivalence class of φ is an element of the topological full group of G acting on $\partial\mathcal{T}$.

Let on the other hand ψ be an element of the topological full group of G acting on $\partial\mathcal{T}$. By definition of the topology on $\partial\mathcal{T}$ and by compactness there exist finitely many vertices $v_1, \dots, v_n \in \mathcal{V}$ and $g_1, \dots, g_n \in G$ such that $\partial\mathcal{T} = \bigsqcup_{i=1}^n \partial\mathcal{T}_{v_i}$ and such that for every i holds $\psi|_{\partial\mathcal{T}_{v_i}} = g_i|_{\partial\mathcal{T}_{v_i}}$. We claim that the subtree T of \mathcal{T} spanned by $\{v_1, \dots, v_n\}$ is finite and complete. It is the union of all images of rooted geodesics to v_i for $i = 1, \dots, n$. Finiteness is therefore obvious. For completeness, assume that there is a vertex v of T which is not a leaf, but such that v has a child w which is not a vertex in \mathcal{T} . Then $\partial\mathcal{T}_w$ is not contained in $\bigcup_{i=1}^n \partial\mathcal{T}_{v_i}$, which is a contradiction. With the same argument the tree T' spanned by $\{g_1(v_1), \dots, g_n(v_n)\}$ is finite

and complete and we can see that ψ can be viewed as a G -honest almost automorphism $\mathcal{T} \setminus T \rightarrow \mathcal{T} \setminus T'$ with $\psi|_{\mathcal{T}_{v_i}} = g|_{\mathcal{T}_{v_i}}$. \square

2.3.1 Topology on almost automorphisms

Let $G \leq \text{Aut}(\mathcal{T})$ be a closed subgroup. In particular G is totally disconnected and locally compact. We want to define a group topology on $\mathcal{F}(G)$ such that $G \leq \mathcal{F}(G)$ is open. This is not always possible, but we will see that we can do it if G has Tits' Independence Property. First we need a lemma from Bourbaki.

Lemma 2.22 ([Bou95], Ch. III, Sect. I, Subsect. 2, Prop. 1). *Let Λ be a group and \mathcal{B} be a filter on Λ satisfying the following three conditions.*

1. *For every $U \in \mathcal{B}$ there exists a $V \in \mathcal{B}$ such that $VV \subset U$.*
2. *For every $U \in \mathcal{B}$ holds $U^{-1} \in \mathcal{B}$.*
3. *For every $g \in \Lambda$ and every $V \in \mathcal{B}$ holds $gVg^{-1} \in \mathcal{B}$.*

Then, there exists a unique group topology on Λ such that \mathcal{B} is a neighbourhood basis of the identity element.

Recall the important theorem of Van Dantzig about totally disconnected locally compact groups.

Theorem 2.23 ([VD36], TG. 39). *For every totally disconnected locally compact group the set of its compact open subgroups is a neighbourhood basis of the identity.*

Proposition 2.24. *Assume $G \leq \text{Aut}(\mathcal{T})$ is closed and has Tits' Independence Property. Then there exists a unique group topology on $\mathcal{F}(G)$ such that $G \leq \mathcal{F}(G)$ is an open subgroup.*

Proof. Since G is totally disconnected and locally compact, any group topology on $\mathcal{F}(G)$ for which $G \leq \mathcal{F}(G)$ is open also has to be totally disconnected and locally compact. So we choose the filter on $\mathcal{F}(G)$ defined by

$$\mathcal{B} = \{U \subset \mathcal{F}(G) \mid U \cap G \subset G \text{ contains an open neighbourhood of } id\}.$$

Conditions 1 and 2 of Lemma 2.22 are clearly fulfilled by van Dantzig's Theorem.

To verify Condition 3 let $\varphi \in \mathcal{F}(G)$ and $O \in \mathcal{B}$ be arbitrary. Take a finite complete subtree $T \subset \mathcal{T}$ such that $\text{Fix}_G(T) \subset O$. By Remark 2.15 we can choose T big enough such that there exists a finite complete subtree $T' \subset \mathcal{T}$ and a representative $\varphi: \mathcal{T} \setminus T' \rightarrow \mathcal{T} \setminus T$ as G -honest almost automorphism. Let $g \in \text{Fix}_G(T')$ be arbitrary. Then the G -honest almost automorphism $\varphi g \varphi^{-1}: \mathcal{T} \setminus T \rightarrow \mathcal{T} \setminus T$ fixes the leaves of T and therefore extends to an

element in $\text{Fix}_{\mathcal{F}(G) \cap \text{Aut}(\mathcal{T})}(T)$. Also, on each connected component of $\mathcal{T} \setminus T$ it coincides with an element of G . Hence by Tits' Independence Property it is an element of $\text{Fix}_G(T) \subset O$. We thus proved that $\varphi \text{Fix}_G(T')\varphi^{-1} \subset O$, or equivalently $\text{Fix}_G(T') \subset \varphi^{-1}O\varphi$, which shows $\varphi^{-1}O\varphi \in \mathcal{B}$. \square

2.4 Higman-Thompson groups

We follow the approach of Caprace and De Medts described in [CDM11] Section 6.3. For a general reference with proofs, see Higman's lecture notes [Hig74]. Another good introduction is [Bro87], Section 4.

Definition 2.25. A *plane order* on a rooted tree is a collection of total orders $\{<_v \mid v \in \mathcal{V}\}$ such that for each $v \in \mathcal{V}$ the element $<_v$ is a total order on the children of v .

Remark 2.26. This is called a plane order because it indicates an embedding of the tree into \mathbb{R}^2 with the following properties. The root is at the origin and the children of each vertex are below its parent, arranged from left to right according to the plane order.

Definition 2.27. An almost automorphism $\varphi \in \mathcal{N}$ is called *locally order-preserving* if there exist finite complete subtrees $T_1, T_2 \subset \mathcal{T}$ and a representative as honest almost automorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ satisfying the following. For every vertex v of $\mathcal{T} \setminus T_1$ the restriction of φ on the children of v is order-preserving.

Definition 2.28. The locally order-preserving elements form a subgroup of \mathcal{N} , called the *Higman-Thompson group* $V_{d,d+1}$.

Remark 2.29. Let $T_1, T_2 \subset \mathcal{T}$ be finite complete subtrees with the same number of leaves. Let $\kappa: \mathcal{LT}_1 \rightarrow \mathcal{LT}_2$ be a bijection. Then there exists a unique honest almost automorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ extending κ in an order-preserving way. We call its equivalence class the element of $V_{d,d+1}$ *induced by κ* . Let T'_1 be a finite complete subtree of \mathcal{T} containing T_1 and let T'_2 be the finite complete subtree of \mathcal{T} spanned by $\varphi(\mathcal{LT}'_1)$. Then φ is also induced by the bijection $\varphi|_{\mathcal{LT}'_1}: \mathcal{LT}'_1 \rightarrow \mathcal{LT}'_2$.

Remark 2.30. The definition of $V_{d,d+1}$ depends on the plane order on \mathcal{T} . A new root and a new plane order will give a different Higman-Thompson group, but it will be conjugate to $V_{d,d+1}$ via an element $g \in \text{Aut}(\mathcal{T})$. Explicitly $g \in \text{Aut}(\mathcal{T})$ can be chosen to be the unique element such that $g(v_0)$ is the new root, and after endowing the domain of g with the old and the image of g with the new plane order, for every $v \in \mathcal{V}$ the restriction of g to the children of v is order-preserving.

In Section 3.1 we will specify a plane order giving us a copy of the Higman-Thompson group which will turn out to be useful for our purposes.

Remark 2.31. The $d + 1$ in the second index of $V_{d,d+1}$ refers to the fact that the root v_0 has valency $d + 1$. If the valency of v_0 were k , Definition 2.28 would yield the Higman-Thompson group $V_{d,k}$.

Notation 2.32. For any group Λ , we denote by $D(\Lambda)$ its commutator subgroup, also called the derived subgroup.

Abelianization of $V_{d,d+1}$. Higman proved that $V_{d,k}$ is finitely presented and that it is simple if d is even and has a simple subgroup of index 2, its commutator subgroup $D(V_{d,k})$, if d is odd. We will now describe the quotient map $V_{d,d+1} \rightarrow V_{d,d+1}/D(V_{d,d+1})$. We will not directly need this quotient map for the present work, but we will generalize the concept in Section 4 and therefore present the idea here. First we need to extend the plane order on \mathcal{T} to a total order on \mathcal{V} .

Definition 2.33. Let $\{<_v \mid v \in \mathcal{V}\}$ be a plane order on \mathcal{T} . The *lexographical order* on \mathcal{T} is the total order $<$ on \mathcal{V} defined as follows. The choice of the root v_0 induces a partial order \prec on \mathcal{V} , namely $v \prec w$ if and only if $\mathcal{T}_v \subset \mathcal{T}_w$. Note that \prec is a join-semilattice, i.e. every finite set has a supremum.

- If $v_1, v_2 \in \mathcal{V}$ are such that $v_1 \prec v_2$, then $v_1 < v_2$.
- Otherwise, let v be the supremum of v_1 and v_2 with respect to \prec . For $i = 1, 2$ let v'_i be the child of v satisfying $v_i \prec v'_i$. Then $v_1 < v_2$ if and only if $v'_1 <_v v'_2$.

Let $T_1, T_2, \varphi, \kappa$ as in Definition 2.27. Let $\iota: \mathcal{LT}_2 \rightarrow \mathcal{LT}_1$ be the unique order-preserving bijection with respect to the lexographical order on \mathcal{V} . Then $\iota \circ \kappa$ is a permutation of the elements in \mathcal{LT}_1 and we can consider its sign $\text{sgn}(\iota \circ \kappa) \in \{1, -1\}$. We want to know when the map $\varphi \mapsto \text{sgn}(\iota \circ \kappa)$ descends to a well-defined homomorphism $V_{d,d+1} \rightarrow \{1, -1\}$.

We replace T_1 (and thus also T_2) by a simple expansion T'_1 (respectively T'_2). We denote by κ' the bijection $\mathcal{LT}'_1 \rightarrow \mathcal{LT}'_2$ induced by φ and by $\iota': \mathcal{LT}'_2 \rightarrow \mathcal{LT}'_1$ the unique order-preserving bijection with respect to the lexographical order on \mathcal{V} . If d is odd, then $\text{sgn}(\iota' \circ \kappa') = \text{sgn}(\iota \circ \kappa)$. Therefore the map $\varphi \mapsto \text{sgn}(\iota \circ \kappa)$ descends to a well-defined homomorphism $V_{d,d+1} \rightarrow \{1, -1\}$. The kernel of this homomorphism is the subgroup $D(V_{d,d+1})$, which is simple due to Higman. If d is even, however, it is not difficult to see that $\text{sgn}(\iota' \circ \kappa') = 1$. Therefore the map $\varphi \mapsto \text{sgn}(\iota \circ \kappa)$ does not descend to a well-defined homomorphism $V_{d,d+1} \rightarrow \{1, -1\}$, but $V_{d,d+1}$ is simple, see [Hig74].

3 Higman-Thompson-like groups

Let $F \leq \text{Sym}(D)$ be any subgroup.

Notation 3.1. From now on we will write $\mathcal{N}_F := \mathcal{F}(U(F))$.

In this section we investigate a certain subgroup $V_F \leq \mathcal{N}_F$ that plays a role analogous to the Higman-Thompson group inside Neretin's group. We show that it is isomorphic to the topological full group of a shift of finite type. Consequently, by results of Matui [Mat15], it is finitely presented and its commutator subgroup $D(V_F)$ is simple.

Notation 3.2. We denote the orbits of F by $D^{(0)}, \dots, D^{(l)} \subset D$. For each $i = 0, \dots, l$ we write $d^{(i)} := |D^{(i)}|$.

3.1 A plane order on \mathcal{T}

Recall from Subsection 2.4 that the definition of the Higman-Thompson group $V_{d,d+1}$ as a subgroup of \mathcal{N} depends on the choice of a plane order on \mathcal{V} . Also recall that an element of $V_{d,d+1}$ is induced by two finite complete subtrees T_1, T_2 of \mathcal{T} and a bijection $\mathcal{L}T_1 \rightarrow \mathcal{L}T_2$.

Let T_1, T_2 be finite complete subtrees of \mathcal{T} and let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ be an arbitrary $U(F)$ -honest almost automorphism. Then $\varphi|_{\mathcal{L}T_1}: \mathcal{L}T_1 \rightarrow \mathcal{L}T_2$ is a bijection such that for every $v \in \mathcal{L}T_1$ the colours $\text{col}(v)$ and $\text{col}(\varphi(v))$ are in the same orbit of F . This motivates the following definition.

Definition 3.3. Define V_F to be the subset of $V_{d,d+1}$ consisting of all almost automorphisms admitting a representative φ of the following form. There exist finite complete subtrees T_1, T_2 of \mathcal{T} such that the honest almost automorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ satisfies

- $\varphi \in V_{d,d+1}$ is induced by the bijection $\varphi|_{\mathcal{L}T_1}: \mathcal{L}T_1 \rightarrow \mathcal{L}T_2$ and
- φ has the property that for every $v \in \mathcal{L}T_1$ the colours $\text{col}(v)$ and $\text{col}(\varphi(v))$ lie in the same orbit of F .

In general V_F is not a subgroup of \mathcal{N} , but in the next proposition we construct a plane order on \mathcal{T} for which it is even a very interesting one.

Proposition 3.4. *There exists a plane order on \mathcal{T} with $V_F = V_{d,d+1} \cap \mathcal{N}_F$. More precisely, there exists a plane order on \mathcal{T} such that every honest almost automorphism as in Definition 3.3 is an $U(F)$ -honest almost automorphism.*

Proof. The inclusion $V_F \supset V_{d,d+1} \cap \mathcal{N}_F$ holds independently of the plane order. Let $\psi \in V_{d,d+1} \cap \mathcal{N}_F$. Then there exist finite complete subtrees $T_1, T_2 \subset \mathcal{T}$ such that $\psi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ is an $U(F)$ -almost automorphism which is an element of $V_{d,d+1}$ induced by the bijection $\psi|_{\mathcal{L}T_1}: \mathcal{L}T_1 \rightarrow \mathcal{L}T_2$. We have to show that this bijection has the following property. For all $v \in \mathcal{L}T_1$

an element $g \in U(F)$ such that $\varphi|_{\mathcal{T}_v} = g|_{\mathcal{T}_v}$ satisfies that $\text{col}(v)$ and $\text{col}(gv)$ are in the same orbit of F . This is true because $\text{prm}_{g,v}(\text{col}(v)) = \text{col}(gv)$ by definition.

For the reverse inclusion we first specify a plane order on \mathcal{T} , i.e. a total order $<_v$ on the children of every vertex $v \in \mathcal{V}$, and then prove that it has the desired property.

For the children of v_0 we say that $v <_{v_0} w$ if and only if $\text{col}(v) < \text{col}(w)$. For all other vertices $v \in \mathcal{V}$ the order $<_v$ is determined by $\text{col}(v)$. Choose for each $i = 0, \dots, l$ a colour $\chi^{(i)} \in D^{(i)}$. Let v be any vertex such that $\text{col}(v) = \chi^{(i)}$ and let v_1, v_2 be children of v . Say $v_1 <_v v_2$ if and only if $\text{col}(v_1) < \text{col}(v_2)$. Let now $\chi \in D \setminus \{\chi^{(0)}, \dots, \chi^{(l)}\}$ and let i be such that $\chi \in D^{(i)}$. Choose an element $f_\chi \in F$ with $f_\chi(\chi) = \chi^{(i)}$. Then f_χ restricts to a bijection $D \setminus \{\chi\} \rightarrow D \setminus \{\chi^{(i)}\}$. Let v be any vertex with $\text{col}(v) = \chi$ and let v_1, v_2 be children of v . Say $v_1 <_v v_2$ if and only if $f_\chi(\text{col}(v_1)) < f_\chi(\text{col}(v_2))$.

To verify that this order has the required property, we first prove that $V_F \subset V_{d,d+1} \cap \mathcal{N}_F$. Let T_1, T_2, φ be as in Definition 3.3. Let v be a vertex of $\mathcal{T} \setminus T_1$. Let $g \in \text{Aut}(\mathcal{T})$ be such that $\psi|_{\mathcal{T}_v} = g|_{\mathcal{T}_v}$. We have to prove that $\text{prm}_{g,v} \in F$. We do this in two steps.

Step 1: The vertex v is a leaf of T_1 .

If there is an i with $\text{col}(gv) = \chi^{(i)} \in \{\chi^{(0)}, \dots, \chi^{(l)}\}$, then by construction $\text{prm}_{g,v} = f_{\text{col}(v)} \in F$. If there is an i with $\text{col}(v) = \chi^{(i)} \in \{\chi^{(0)}, \dots, \chi^{(l)}\}$ then $\varphi^{-1}: \mathcal{T} \setminus T_2 \rightarrow \mathcal{T} \setminus T_1$ coincides with g^{-1} on $\mathcal{T}_{\varphi(v)}$, and by Lemma 2.6 and above argument holds $\text{prm}_{g,v} = \text{prm}_{g^{-1},gv}^{-1} = f_{\text{col}(gv)}^{-1} \in F$. Otherwise by Lemma 2.9 there exists an $h \in U(F)$ such that $\text{col}(hv) = \chi^{(i)}$ and $\text{prm}_{h,v} = f_{\text{col}(v)}$. Then Lemma 2.6 yields $\text{prm}_{g,v} = \text{prm}_{gh^{-1}h,v} = \text{prm}_{gh^{-1},hv} \circ \text{prm}_{h,v} = f_{\text{col}(gv)}^{-1} \circ f_{\text{col}(v)} \in F$.

Step 2: The vertex v is a child of a leaf of T_1 .

Let T'_1 denote the simple expansion of T_1 such that $v \in \mathcal{L}T'_1$. Similarly denote by T'_2 the simple expansion of T_2 such that $gv \in \mathcal{L}T'_2$. From Step 1 follows in particular that for all leaves w of T'_1 the colours $\text{col}(w)$ and $\text{col}(gw)$ lie in the same orbit of F . Recall that $\varphi \in V_{d,d+1}$ is also induced by the bijection $\varphi|_{\mathcal{L}T'_1}: \mathcal{L}T'_1 \rightarrow \mathcal{L}T'_2$. Thus $T'_1, T'_2, \varphi|_{\mathcal{T} \setminus T'_1}$ satisfies the conditions from Definition 3.3. Now we can repeat the argument from Step 1 for φ replaced by $\varphi|_{\mathcal{T} \setminus T'_1}$ and get that $\text{prm}_{g,v} \in F$.

Recall that every finite complete subtree containing T_1 is obtained from T_1 by a finite sequence of simple expansions. Therefore we iteratively get that for all vertices v of $\mathcal{T} \setminus T_1$ holds $\text{prm}_{g,v} \in F$. This means that indeed $V_F \subset V_{d,d+1} \cap \mathcal{N}_F$. \square

Example 3.5. Consider the 4-regular tree from Figure 4. Its plane order is implied by how the tree is drawn from left to right, namely, for any children v_1, v_2 of a vertex v holds $v_1 <_v v_2$ if and only if v_1 is drawn to the left of v_2 .

Let $F = \langle (12) \rangle \leq S_4$. Let $T_1 = T_2$ be the finite complete subtree whose

leaves are the children of v_0 . Then there exists an element in \mathcal{N}_F (even in $U(F)$) switching the leaves of colour 1 and 2. Note, however, that the element of $V_{d,d+1}$ induced by this permutation of \mathcal{LT}_1 is not an element of \mathcal{N}_F .

Let now $F = \langle (123) \rangle$. Then $V_F = V_{d,d+1} \cap \mathcal{N}_F$ holds with the drawn order.

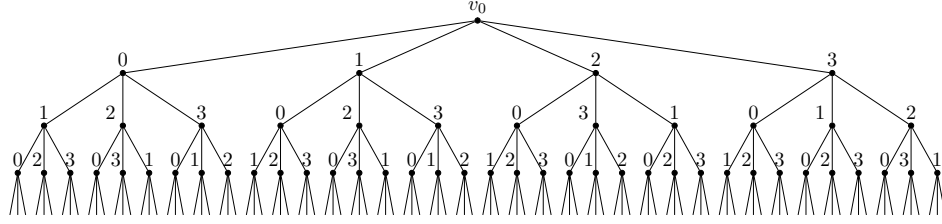


Figure 4: The left-to-right drawing specifies an order on the tree as in Proposition 3.4 for $F = \langle (1, 2, 3) \rangle < S_4$.

3.2 The group V_F as topological full group

For the rest of the article we will assume that the plane order on \mathcal{T} is as in Proposition 3.4.

3.2.1 The topological full group of a groupoid

The topological full group of a groupoid was defined and studied by Matui, e.g. in [Mat12]. We refer for the preliminaries there for a more detailed introduction to it.

Topological groupoids. A *groupoid* is a category such that every morphism is an isomorphism. For our purposes we assume in addition that it is a small category, i.e. the class of objects as well as the class of morphisms are sets. A *topological groupoid* is a groupoid \mathcal{G} such that the set of objects and the set of morphisms are topological spaces and all structure maps (composition, inverse, identity, source and range) are continuous. We denote by $\mathcal{G}^{(0)}$ the space of objects and by $\mathcal{G}^{(1)}$ the space of morphisms of \mathcal{G} . Denote by

$$s, r: \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(0)}$$

the source and range maps. A topological groupoid is called *étale* if s and r are local homeomorphisms.

Definition 3.6. Let \mathcal{G} be an étale groupoid. A *bisection* of \mathcal{G} is a clopen subset $U \subset \mathcal{G}^{(1)}$ such that $s|_U: U \rightarrow \mathcal{G}^{(0)}$ and $r|_U: U \rightarrow \mathcal{G}^{(0)}$ are homeomorphisms.

Definition 3.7. The *topological full group* of an étale groupoid \mathcal{G} is

$$[[\mathcal{G}]] := \{r \circ (s|_U)^{-1} \in \text{Homeo}(\mathcal{G}^{(0)}) \mid U \subset \mathcal{G}^{(1)} \text{ bisection of } \mathcal{G}\}.$$

We leave to the reader to check that it is indeed a subgroup of $\text{Homeo}(\mathcal{G}^{(0)})$.

Recall that in Definition 2.20 we already had the notion of a topological full group, namely of a group acting on a topological space. We will now explain how this fits into the framework of groupoids.

Example 3.8. Let Λ be a discrete group acting by homeomorphisms on a Cantor space X . The *action groupoid* for this group action is defined as follows. The set of objects is X . The space of morphisms is $\Lambda \times X$ with the product topology. Source and range are given by $s(g, x) = x$ and $r(g, x) = gx$. Since Λ is discrete s and r are local homeomorphisms. Two morphisms (g, x) and (h, y) are thus composable if and only if $x = hy$ and the product is then given by $(g, hy) \cdot (h, y) = (gh, y)$. A bisection is a disjoint union

$$U := \bigsqcup_{i=1}^n \{g_i\} \times U_i \subset \Lambda \times X$$

such that $\{U_i \mid 1 \leq i \leq n\}$ is a clopen partition of X and such that the map $\varphi: X \rightarrow X$ satisfying $\varphi|_{U_i}: U_i \rightarrow X, x \mapsto g_i x$ is a homeomorphism. Then $\varphi = r \circ (s|_U)^{-1}$ is an element of the topological full group. We can thus see that the topological full group of the action groupoid is the same as the topological full group of the group action as in Definition 2.20.

3.2.2 One-sided shifts of finite type

We refer to [Mat15], Section 6, for a more detailed treatment of shifts of finite type and the topological full group associated to them.

Definition 3.9. Let $\mathfrak{g} = (V, E)$ be an oriented graph and $E' \subset E$ its orientation. The *adjacency matrix* of \mathfrak{g} is the matrix $M_{\mathfrak{g}} \in \mathbb{Z}^{V \times V}$ such that $M_{\mathfrak{g}}(v, w) = |\{e \in E' \mid i(e) = v, t(e) = w\}|$ for all $v, w \in V$.

Assumptions on oriented graphs. Let $\mathfrak{g} = (V, E)$ be an oriented graph and $M_{\mathfrak{g}}$ its adjacency matrix. We will always require two conditions on \mathfrak{g} respectively $M_{\mathfrak{g}}$. The first condition is that it must be irreducible, i.e. for all $v, w \in V$ there exists an n such that $M_{\mathfrak{g}}^n(v, w) \neq 0$. This is equivalent to saying that there exists a path of length n from v to w . The second condition is that $M_{\mathfrak{g}}$ must not be a permutation matrix, which is equivalent to saying that \mathfrak{g} is not a disjoint union of oriented cycles.

One-sided irreducible shifts of finite type. Let

$$X_{\mathfrak{g}} = \{(e_k) \in (E')^{\mathbb{N}} \mid i(e_{k+1}) = t(e_k)\}$$

be the set of infinite oriented paths in \mathfrak{g} . Note that $X_{\mathfrak{g}} \subset (E')^{\mathbb{N}}$ is closed. Moreover, that $M_{\mathfrak{g}}$ is irreducible and not a permutation matrix ensures that $X_{\mathfrak{g}}$ is a Cantor space. Define the map $\sigma: X_{\mathfrak{g}} \rightarrow X_{\mathfrak{g}}$ by $\sigma(e)_k = e_{k+1}$. It is a local homeomorphism. The pair $(X_{\mathfrak{g}}, \sigma)$ is called the *one-sided irreducible shift of finite type associated to \mathfrak{g}* .

Associated groupoid. We associate to $(X_{\mathfrak{g}}, \sigma)$ the following groupoid $\mathcal{G}_{\mathfrak{g}}$. The space of objects and morphisms are

$$\begin{aligned} \mathcal{G}_{\mathfrak{g}}^{(0)} &= X_{\mathfrak{g}} \\ \mathcal{G}_{\mathfrak{g}}^{(1)} &= \{(x, n-m, y) \in X_{\mathfrak{g}} \times \mathbb{Z} \times X_{\mathfrak{g}} \mid \sigma^n(x) = \sigma^m(y)\} \subset X_{\mathfrak{g}} \times \mathbb{Z} \times X_{\mathfrak{g}}. \end{aligned}$$

We endow $\mathcal{G}_{\mathfrak{g}}^{(1)}$ with the topology that is generated by all sets of the form $\{(x, n-m, y) \mid x \in U, y \in V, \sigma^n(x) = \sigma^m(y)\}$ with $U, V \subset X_{\mathfrak{g}}$ clopen. The source and range maps are the projection on the first respectively last factor. Two elements $(x, n-m, y)$ and $(y', m-l, z)$ are composable if and only if $y = y'$ and the product is $(x, n-m, y) \cdot (y, m-l, z) = (x, n-l, z)$. The unit space consists of all elements of the form $(x, 0, x)$ and is homeomorphic to $X_{\mathfrak{g}}$ in an obvious way. The inverse is given by $(x, n-m, y)^{-1} = (y, m-n, x)$.

Theorem 3.10 ([Mat15], Section 6). *Let \mathfrak{g} be a finite oriented graph such that the associated adjacency matrix $M_{\mathfrak{g}}$ is irreducible and not a permutation matrix. Then, the topological full group $[[\mathcal{G}_{\mathfrak{g}}]]$ is finitely presented (more precisely, it is of type F_{∞}). Moreover, every non-trivial subgroup of $[[\mathcal{G}_{\mathfrak{g}}]]$ normalized by $D([[\mathcal{G}_{\mathfrak{g}}]])$ contains $D([[\mathcal{G}_{\mathfrak{g}}]])$. In particular $D([[\mathcal{G}_{\mathfrak{g}}]])$ is simple. Its abelianization is isomorphic to*

$$[[\mathcal{G}_{\mathfrak{g}}]]/D([[\mathcal{G}_{\mathfrak{g}}]]) \cong (\text{Coker}(id - M_{\mathfrak{g}}^t) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \text{Ker}(id - M_{\mathfrak{g}}^t).$$

3.2.3 A finite oriented graph associated to $U(F)$.

We will now construct the finite oriented graph such that the topological full group of the associated groupoid is isomorphic to V_F . Let us first motivate the construction informally before defining the graph explicitly and proving that it has the desired property.

Motivation. We denote by \mathfrak{g}_F the graph we are constructing. The group V_F acts by homeomorphisms on $\partial\mathcal{T}$, which is a set of infinite paths. Similarly the group $[[\mathcal{G}_{\mathfrak{g}_F}]]$ acts by homeomorphisms on $X_{\mathfrak{g}_F}$, which is also a set of infinite paths. We want to identify $\partial\mathcal{T}$ and $X_{\mathfrak{g}_F}$ in a way that identifies V_F and $[[\mathcal{G}_{\mathfrak{g}_F}]]$.

Let $v \neq v_0$ be a vertex of \mathcal{T} . Consider the finite rooted geodesic with endpoint v . Since v has d children, the path has d possibilities to continue. Let i be such that $\text{col}(v) \in D^{(i)}$. Then $d^{(i)} - 1$ of the d possibilities to continue are by going along an edge of colour in $D^{(i)}$. Likewise for every $j \neq i$ exactly $d^{(j)}$ of the d possibilities to continue are along an edge of colour in $D^{(j)}$.

The group V_F does not distinguish between different colours, only between colour orbits. We want to see this in \mathbf{g}_F . For every $i = 0, \dots, l$ we say the graph \mathbf{g}_F has one vertex which we can identify with $D^{(i)}$. Imitating the fact that the vertex v above has d children we want that every vertex of \mathbf{g}_F has d outgoing edges. From the d outgoing edges of $D^{(i)}$ we let $d^{(i)} - 1$ end back in $D^{(i)}$ and for every $j \neq i$ we let $d^{(j)}$ edges end in $D^{(j)}$.

This graph now almost does what we want, but there is still a problem. Namely the root v_0 has $d + 1$ children, which means a path beginning at v_0 in \mathcal{T} has $d + 1$ options how it can start. Of those, $d^{(i)}$ are along an edge of colour in $D^{(i)}$ for every i . But the graph \mathbf{g}_F so far only has $l + 1$ vertices, which means a path in \mathbf{g}_F only has $l + 1$ options to start. We solve this problem by adding one additional vertex on each of the loops at $D^{(i)}$ for every $i = 0, \dots, l$. Since these newly added vertices have exactly one ingoing and one outgoing edge, they have no influence on the path space $X_{\mathbf{g}_F}$. A path starting at such a vertex must continue to $D^{(i)}$, so we can think of it as starting at $D^{(i)}$ and say that there are $d^{(i)}$ options for a path to start at $D^{(i)}$ for every $i = 0, \dots, l$.

The graph \mathbf{g}_F . More formally, define the oriented graph $\mathbf{g}_F = (\mathfrak{V}_F, \mathfrak{E}_F)$ with orientation $\mathfrak{E}'_F \subset \mathfrak{E}_F$ as follows. It is enough to specify \mathfrak{V}_F and the adjacency matrix $M_{\mathbf{g}_F}$. Denote $\mathcal{D} := \{D^{(0)}, \dots, D^{(l)}\}$. We define the vertex set to be the union $\mathfrak{V}_F = \mathcal{D} \cup \bigcup_{i=0}^l \{\delta_1^{(i)}, \dots, \delta_{d^{(i)}-1}^{(i)}\}$. The adjacency matrix in $\mathbb{Z}^{\mathfrak{V}_F \times \mathfrak{V}_F}$ is defined to be

$$M_{\mathbf{g}_F}(v, w) = \begin{cases} d^{(i)} & v = D^{(j)}, w = D^{(i)}, i \neq j \\ 1 & v = D^{(i)}, w = \delta_1^{(i)}, \dots, \delta_{d^{(i)}-1}^{(i)} \\ 1 & v = \delta_1^{(i)}, \dots, \delta_{d^{(i)}-1}^{(i)}, w = D^{(i)} \\ 0 & \text{else.} \end{cases}$$

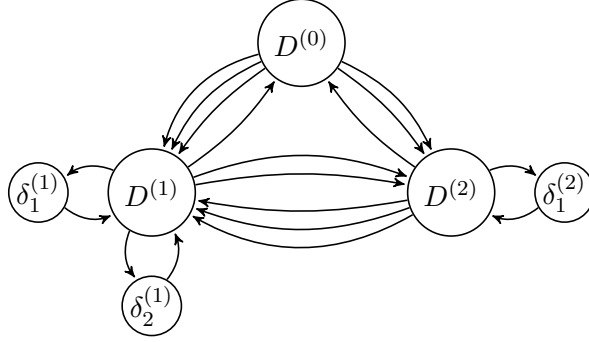


Figure 5: The graph \mathbf{g}_F for $F < S_6$ with $(d^{(0)}, d^{(1)}, d^{(2)}) = (1, 3, 2)$

Theorem 3.11. *The groups V_F and $[[\mathcal{G}_{\mathbf{g}_F}]]$ are isomorphic.*

Proof. Denote by $\mathcal{P}(\mathbf{g}_F)$ the set of finite oriented paths in \mathbf{g}_F with terminal point in \mathcal{D} . The outline of the proof is the following. We first construct a $(d+1)$ -regular tree $\tilde{\mathcal{T}}$ with vertex set $\mathcal{P}(\mathbf{g}_F) \cup \{\emptyset\}$, where we consider \emptyset the root. The set of infinite paths $X_{\mathbf{g}_F}$ can be identified with the set of infinite paths $\partial\tilde{\mathcal{T}}$ in an obvious way. We endow $\tilde{\mathcal{T}}$ with a plane order. Then there exists a unique tree isomorphism $\omega: \tilde{\mathcal{T}} \rightarrow \mathcal{T}$ with $\omega(\emptyset) = v_0$ and preserving the order on the children of every vertex. The plane order on $\tilde{\mathcal{T}}$ has the property that for all $\tilde{e} \in \mathcal{P}(\mathbf{g}_F)$ holds $\text{col}(\omega(\tilde{e})) \in t(\tilde{e})$, where $t(\tilde{e})$ denotes the terminal vertex of the oriented path \tilde{e} . We will see that the elements of $[[\mathcal{G}_{\mathbf{g}_F}]]$ can be written as precisely those almost automorphisms of $\tilde{\mathcal{T}}$ which under ω correspond to elements of V_F .

Let $i \in \{0, \dots, l\}$. In the construction of $\tilde{\mathcal{T}}$ we will repeatedly need the set of the shortest possible oriented paths starting at $D^{(i)}$ and ending in \mathcal{D} . We denote it by

$$\Omega^{(i)} = \Omega_1^{(i)} \cup \Omega_2^{(i)},$$

where

$$\Omega_1^{(i)} := \{(e_0) \in \mathcal{P}(\mathbf{g}_F) \mid e_0 \in \mathfrak{E}'_F, i(e_0) = D^{(i)}, t(e_0) \in \mathcal{D}\}$$

is the set of such shortest paths starting at but not ending in $D^{(i)}$ and

$$\Omega_2^{(i)} := \{(e_0, e_1) \in \mathcal{P}(\mathbf{g}_F) \mid e_0, e_1 \in \mathfrak{E}'_F, i(e_0) = D^{(i)}, t(e_0) \notin \mathcal{D}\}$$

is the set of such shortest paths starting at and ending in $D^{(i)}$. Note that $|\Omega_2^{(i)}| = d^{(i)} - 1$ and $|\Omega_1^{(i)}| = \sum_{j \neq i} d^{(j)}$.

Now we are ready to construct $\tilde{\mathcal{T}}$. As mentioned we declare \emptyset to be the root of $\tilde{\mathcal{T}}$. We define the children of \emptyset to be the paths of minimal lengths ending in \mathcal{D} , i.e. the elements of \mathcal{D} and the edges $e' \in \mathfrak{E}'$ with $o(e') = \delta_j^{(i)}$. Now iteratively for every $\tilde{e} \in \mathcal{P}(\mathbf{g}_F)$ with $t(\tilde{e}) = D^{(i)}$ the set of children of \tilde{e} is

$$\{(\tilde{e}, e) \mid e \in \Omega^{(i)}\}.$$

Informally speaking every child of the path \tilde{e} is \tilde{e} continued one step further.

Next we define a plane order on $\tilde{\mathcal{T}}$, i.e. an order on the children of every vertex. Note that the root \emptyset has for every $i = 0, \dots, l$ exactly $d^{(i)}$ children with endpoint $D^{(i)}$. Therefore there exists an order on the children of \emptyset such that the order-preserving bijection η from the children of \emptyset to the children of v_0 satisfies that for every child \tilde{v} of \emptyset holds $\text{col}(\eta(\tilde{v})) \in t(\tilde{v})$. Endow the set of children of \emptyset with such an order. To define an order on the children of the other vertices, we need a little preparation. For every $i = 0, \dots, l$ fix an arbitrary vertex $v^{(i)} \in \mathcal{V}$ with $\text{col}(v^{(i)}) \in D^{(i)}$. Further choose a bijection $\zeta^{(i)}$ from $\Omega^{(i)}$ to the children of $v^{(i)}$ with the property that $\text{col}(\zeta^{(i)}(e)) \in t(e)$ for every $e \in \Omega^{(i)}$. Informally speaking this means we identify the $d^{(j)}$ edges from $D^{(i)} \in V$ to $D^{(j)} \in V$ with the elements of $D^{(j)} \subset D$ and we identify the $d^{(i)} - 1$ interrupted loops at $D^{(i)}$ with the elements of $D^{(i)} \setminus \{\text{col}(v^{(i)})\}$. For every vertex \tilde{e} of $\tilde{\mathcal{T}}$ which is not the root its set of children has the form $\{(\tilde{e}, e) \mid e \in \Omega^{(i)}\}$, where i is such that $t(\tilde{e}) = D^{(i)}$. Let now $e, e' \in \Omega^{(i)}$. We say that $(\tilde{e}, e) < (\tilde{e}, e')$ if and only if $\zeta^{(i)}(e) < \zeta^{(i)}(e')$.

Now we want to show that elements of $[[\mathcal{G}_{\mathbf{g}_F}]]$ are locally order-preserving almost automorphisms of $\tilde{\mathcal{T}}$. There is an obvious identification between the set of infinite paths $X_{\mathbf{g}_F}$ and the boundary $\partial\tilde{\mathcal{T}}$. Let $U \subset \mathcal{G}_{\mathbf{g}_F}$ be a bisection. By definition there exist clopen partitions $\{U_1, \dots, U_n\}$ and $\{U'_1, \dots, U'_n\}$ of $X_{\mathbf{g}_F}$ and positive integers $n_1, \dots, n_n, m_1, \dots, m_n$ satisfying the following. The bisection U can be written as

$$U = \bigsqcup_{k=1}^n U_k \times \{n_k - m_k\} \times U'_k$$

and $r \circ (s|_U)^{-1}$ restricts to a homeomorphism $U_k \rightarrow U'_k$ for every $k = 1, \dots, n$. By making the U_i and U'_i smaller if necessary, we can assume that for every $k = 1, \dots, n$ there exist finite paths $(e_{k0}, \dots, e_{kn_k})$ and $(e'_{k0}, \dots, e'_{km_k})$ such that

$$\begin{aligned} U_k &= \{(e_i)_{i \in \mathbb{N}} \in X \mid \forall 0 \leq i \leq n_k: e_i = e_{ki}\} \\ U'_k &= \{(e'_i)_{i \in \mathbb{N}} \in X \mid \forall 0 \leq i \leq m_k: e'_i = e'_{ki}\} \end{aligned}$$

and

$$r \circ (s|_U)^{-1}(e_{k0}, \dots, e_{kn_k}, e_{n_k+1}, e_{n_k+2}, \dots) = (e'_{k0}, \dots, e'_{km_k}, e_{n_k+1}, \dots).$$

Note that since the vertices of \mathbf{g}_F of the form $\delta_j^{(i)}$ have precisely one outgoing edge we can assume that $t(e_{kn_k}), t(e'_{kn_k}) \in \mathcal{D}$ for all $k = 1, \dots, n$ without changing U_k and U'_k . This means that $(e_{k0}, \dots, e_{kn_k})$ and $(e'_{k0}, \dots, e'_{kn_k})$ are vertices of $\tilde{\mathcal{T}}$. Observe that for $k = 0, \dots, n$ holds

$$\begin{aligned} U_k &= \partial\tilde{\mathcal{T}}_{(e_{k0}, \dots, e_{kn_k})} \\ U'_k &= \partial\tilde{\mathcal{T}}_{(e'_{k0}, \dots, e'_{kn_k})}. \end{aligned}$$

Since $\{U_1, \dots, U_n\}$ is a clopen partition of $X_{\mathfrak{g}_F}$, there exists a finite complete subtree $T \subset \tilde{\mathcal{T}}$ with

$$\mathcal{L}T = \{(e_{k0}, \dots, e_{kn_k}) \mid k = 1, \dots, n\}.$$

In the same way there exists a finite complete subtree $T' \subset \tilde{\mathcal{T}}$ with

$$\mathcal{L}T' = \{(e'_{k0}, \dots, e'_{km_k}) \mid k = 1, \dots, n\}.$$

The element $r \circ (s|_U)^{-1}$ is then an almost automorphism $\tilde{\mathcal{T}} \setminus T \rightarrow \tilde{\mathcal{T}} \setminus T'$. It has the property that for every vertex $\tilde{e} = (e_{k0}, \dots, e_{kn_k}, e_{n_k+1}, e_{n_k+2}, \dots, e_{i'})$ of $\tilde{\mathcal{T}} \setminus T$ holds

$$r \circ (s|_U)^{-1}(\tilde{e}) = (e'_{k0}, \dots, e'_{km_k}, e_{i+1}, e_{i+2}, \dots, e_{i'}).$$

Therefore this almost automorphism is locally order-preserving. This proves that every element of $[[\mathcal{G}_{\mathfrak{g}_F}]]$ is a locally order-preserving almost automorphism of $\tilde{\mathcal{T}}$.

In addition, let $(e_{k0}, \dots, e_{kn_k}), (e'_{k0}, \dots, e'_{km_k}) \in \mathcal{P}(\mathfrak{g}_F)$ be finite paths, with $k = 0, \dots, n$. Let

$$\begin{aligned} U_k &= \{(e_i)_{i \in \mathbb{N}} \in X \mid \forall 0 \leq i \leq n_k: e_i = e_{ki}\} \\ U'_k &= \{(e'_i)_{i \in \mathbb{N}} \in X \mid \forall 0 \leq i \leq m_k: e'_i = e'_{ki}\} \end{aligned}$$

be such that $\{U_1, \dots, U_n\}$ and $\{U'_1, \dots, U'_n\}$ are clopen partitions of $X_{\mathfrak{g}_F}$. Then $U = \bigsqcup_{k=1}^n U_k \times \{n_k - m_k\} \times U'_k$ is a bisection if and only if

$$t(e_{k0}, \dots, e_{kn_k}) = t(e'_{k0}, \dots, e'_{km_k})$$

for every $k = 1, \dots, n$.

Let, on the other hand, $T, T' \subset \tilde{\mathcal{T}}$ be finite complete subtrees and let $\varphi: \mathcal{T} \setminus T \rightarrow \mathcal{T} \setminus T'$ be an honest almost automorphism satisfying the following. It is locally order-preserving and for all $\tilde{e} \in \mathcal{L}T$ holds $t(\tilde{e}) = t(\varphi(\tilde{e}))$. Repeating above argument in reverse shows that the sets of leaves $\mathcal{L}T$ and $\mathcal{L}T'$ define two compact and open partitions of $\partial\tilde{\mathcal{T}}$. These partitions together with the bijection $\mathcal{L}T \rightarrow \mathcal{L}T'$ induced by φ can also be interpreted as a bisection of $\mathcal{G}_{\mathfrak{g}_F}$. This gives rise to an element of $[[\mathcal{G}_{\mathfrak{g}_F}]]$.

To conclude the proof recall from Proposition 3.4 that the plane order on \mathcal{T} has the following property. For every finite complete subtrees T_1, T_2 of \mathcal{T} and every bijection $\kappa: \mathcal{L}T_1 \rightarrow \mathcal{L}T_2$ respecting the orbits of the colors of the leaves of T_1 and T_2 , the locally order-preserving almost automorphism induced by κ is an $U(F)$ -honest almost automorphism. Let $\omega: \tilde{\mathcal{T}} \rightarrow \mathcal{T}$ be the unique tree isomorphism with $\omega(\emptyset) = v_0$ and preserving the order on the children on every vertex. It has the following property. By definition of the plane order on $\tilde{\mathcal{T}}$, for every $\tilde{e} \in \mathcal{P}(\mathfrak{g}_F)$ holds $\text{col}(\omega(\tilde{e})) \in t(\tilde{e})$. Therefore, the almost automorphisms in $[[\mathcal{G}_{\mathfrak{g}_F}]]$ and the almost automorphisms in V_F exactly correspond to each other under ω . \square

4 Compact generation and virtual simplicity

Let $F \leq \text{Sym}(D)$ be any subgroup. In this section we prove that \mathcal{N}_F is compactly generated and that $D(\mathcal{N}_F)$ is open, simple and has finite index in \mathcal{N}_F . Compact generation is a direct consequence of the theorem below. The statement is the analog to saying that Neretin's group contains a dense copy of a Higman-Thompson group.

Theorem 4.1. *The finitely generated group V_F is dense in \mathcal{N}_F .*

Proof. Let $T \subset \mathcal{T}$ be an arbitrarily big finite complete subtree. We need to prove that for every element $\varphi \in \mathcal{N}_F$ there exists an element $\psi \in V_F$ such that $\psi^{-1}\varphi \in \text{Fix}_{U(F)}(T)$. Let $T_1, T_2 \subset \mathcal{T}$ be two finite complete subtrees and let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ be an $U(F)$ -honest almost automorphism. By Remark 2.15 we can assume that $T_1 \supset T$. Then φ restricts to a bijection $\mathcal{L}T_1 \rightarrow \mathcal{L}T_2$ which induces an element $\psi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ in $V_{d,d+1}$ such that $\psi|_{\mathcal{L}T_1} = \varphi|_{\mathcal{L}T_1}$. Theorem 3.11 implies that $\psi \in V_F$. In addition we observe that $\psi^{-1}\varphi \in \text{Fix}_{U(F)}(T)$, which concludes the proof. \square

Definition 4.2. Let a group Λ act on a topological space X . This action is called *minimal* if every orbit is dense.

Definition 4.3. Let a group Λ act on the Cantor space X . This action is called *purely infinite* if for every nonempty compact open subset $U \subsetneq X$ there exist $g, h \in \Lambda$ such that $g(U) \cup h(U) \subset U$ and $g(U) \cap h(U) = \emptyset$.

Remark 4.4. Let a group Λ act on a topological space X . Let $\Lambda' < \Lambda$ be a subgroup such that the restriction of the action to Λ' is minimal. Then, the action of Λ is minimal as well.

The analogous statement holds for purely infinite actions.

Theorem 4.5 ([Mat15], Theorem 4.16). *Let a group Λ act minimally on the Cantor space such that the action is purely infinite. The commutator subgroup of the topological full group of this action is simple.*

Remark 4.6. In the article where this theorem is stated, it is assumed that G is countable and the action is essentially free. However, a close inspection of the proof shows that these two assumptions are not used. I am grateful to Hiroki Matui for clarifying this point with me.

Corollary 4.7. *The commutator subgroup $D(\mathcal{N}_F)$ is simple.*

Proof. By Theorem 4.5 it suffices to show that the action $U(F) \curvearrowright \partial\mathcal{T}$ is purely infinite and minimal. Since $U(\{id\}) \leq U(F)$, by Remark 4.4 it suffices to consider $F = \{id\}$.

First we show that the action is purely infinite. Denote by dist the usual distance on \mathcal{T} measuring distance 1 between neighbouring vertices. Let $U \subsetneq \partial\mathcal{T}$ be a clopen subset. Choose $v, w_1, w_2 \in \mathcal{V}$ such that

- a) $\text{col}(w_1) \neq \text{col}(v) \neq \text{col}(w_2)$
- b) $\mathcal{T}_v, \mathcal{T}_{w_1}$ and \mathcal{T}_{w_2} are pairwise disjoint
- c) $\partial\mathcal{T}_v \subset \partial\mathcal{T} \setminus U$ and
- d) $\partial\mathcal{T}_{w_1} \cup \partial\mathcal{T}_{w_2} \subset U$.

For $i = 1, 2$ we want to find elements in $U(\{id\})$ mapping U into $\partial\mathcal{T}_{w_i}$. Let $L_i = (e_{i,j})_{j \in \mathbb{Z}} \subset \mathcal{E}^{\mathbb{Z}}$ be a doubly-infinite geodesic as follows. It is uniquely defined by the conditions

- $o(e_{i,0}) = v$
- $t(e_{i,\text{dist}(v,w_i)-1}) = w_i$
- for $j_1 \equiv j_2 \pmod{\text{dist}(v,w_i)}$ holds $\text{col}(e_{i,j_1}) = \text{col}(e_{i,j_2})$.

Written out in words, this means that the subpath $(e_{i,j})_{0 \leq j \leq \text{dist}(v,w_i)-1}$ of L_i is the geodesic from v to w_i , and L_i is coloured periodically. The geodesic L_i exists because conditions a) and b) imply that $\text{col}(e_{i,0}) \neq \text{col}(e_{i,\text{dist}(v,w_i)-1})$. It defines one point in $\partial\mathcal{T}_{w_i} \subset U$ and one point in $\partial\mathcal{T}_v$, which is disjoint from U , in an obvious way. For every integer $n \geq 0$ there exists a unique translation of translation length $n \cdot \text{dist}(v,w_i)$ along L_i in $U(\{id\})$ with attracting point in $\partial\mathcal{T}_{w_i}$ and repelling point outside of U . For sufficiently large n the image of U under this translation is contained in $\partial\mathcal{T}_{w_i}$.

Now we prove minimality. Assume there exists an orbit in $\partial\mathcal{T}$ which is not dense. Then, its closure is the boundary of a proper $U(\{id\})$ -invariant subtree of \mathcal{T} . This is not possible because $U(\{id\})$ acts transitively on \mathcal{V} . \square

To investigate the abelianization of V_F , we need the well-known Smith normal form. For the reader's convenience we recall the statement here.

Lemma 4.8 (Smith normal form). *Let R be a principal ideal domain and let $M \in R^{m \times n}$ be a matrix. Then, there exist invertible matrices $S \in R^{m \times m}$ and $T \in R^{n \times n}$, an integer $k \leq \min\{m, n\}$ and elements $\epsilon_1, \dots, \epsilon_k \in R$, called elementary divisors, such that*

- ϵ_i is the i -th diagonal entry of SMT ,
- all other entries of SMT are 0 and
- ϵ_i divides ϵ_{i+1} for $k = 1, \dots, k-1$.

The elementary divisors are unique up to multiplication with a unit. They have the property that the product $\epsilon_1 \dots \epsilon_i$ is the greatest common divisor of the determinants of all $i \times i$ -submatrices. Furthermore

$$\text{Coker}(M) \cong R^{m-k} \times \prod_{i=1}^k R/R\epsilon_i.$$

Notation 4.9. As in the preceeding section we denote the orbits of F by $D^{(0)}, \dots, D^{(l)} \subset D$. For each $i = 0, \dots, l$ we write $d^{(i)} := |D^{(i)}|$.

Proposition 4.10. *The commutator subgroup $D(V_F)$ has finite index in V_F . More precisely, if all $d^{(i)}$ are even, the abelianization of V_F is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{l+1}$. Otherwise it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^l$.*

Proof. By Theorem 3.10 the abelianization is isomorphic to

$$V_F/D(V_F) \cong (\text{Coker}(id - M_{\mathfrak{g}_F}^t) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \text{Ker}(id - M_{\mathfrak{g}_F}^t).$$

To determine $\text{Coker}(id - M_{\mathfrak{g}_F}^t)$ and $\text{Ker}(id - M_{\mathfrak{g}_F}^t)$ we use the Smith normal form. When writing out the matrix $id - M_{\mathfrak{g}_F}^t$ explicitly, it is not hard to see that performing elementary row- and column operations on $id - M_{\mathfrak{g}_F}^t$ we get the block diagonal matrix

$$id - M_{\mathfrak{g}_F}^t \sim \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \hline 0 & 0 & \dots & \dots & 0 & 1-d & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 & -d^{(1)} & 2 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 & -d^{(2)} & 0 & 2 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & -d^{(l-1)} & 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & \dots & \dots & 0 & -d^{(l)} & 0 & 0 & \dots & 0 & 2 \end{array} \right),$$

which has a $(d-l) \times (d-l)$ identity matrix in the upper left corner. The determinant of this matrix is $2^l \cdot (1-d)$, therefore $\text{Ker}(id - M_{\mathfrak{g}_F}^t) = \{0\}$. This already implies that the number of elementary divisors is $d+1$, so $\text{Coker}(id - M_{\mathfrak{g}_F}^t)$ is finite and therefore $D(V_F)$ has finite index in V_F .

We now determine the abelianization $V_F/D(V_F)$. Let $\epsilon_1, \dots, \epsilon_{d+1}$ be the elementary divisors of $id - M_{\mathfrak{g}_F}^t$. Lemma 4.8 says

$$\text{Coker}(id - M_{\mathfrak{g}_F}^t) \cong \prod_{i=1}^{d+1} \mathbb{Z}/\epsilon_i \mathbb{Z}.$$

The first $d-l$ elementary divisors are $\epsilon_1 = \dots = \epsilon_{d-l} = 1$. The $(d-l+1)$ -th to $(d+1)$ -th elementary divisors are the elementary divisors of the second of two blocks in the above block diagonal matrix. They are given by the greatest common divisors of determinants of submatrices. Note that all possibly odd matrix entries are in the same column. From that we see that if one of the $d^{(i)}$ is odd, then the $(d-l+1)$ -th elementary divisor

is odd and the further ones are even, otherwise all are even. Now since $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$ and since tensor product is distributive with direct sums, we get that

$$V_F/D(V_F) \cong \prod_{i=1}^{d+1} \mathbb{Z}/\epsilon_i\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{l+1} & \text{if all } d^{(i)} \text{ are even} \\ (\mathbb{Z}/2\mathbb{Z})^l & \text{otherwise.} \end{cases}$$

□

Theorem 4.11. *The commutator subgroup $D(\mathcal{N}_F)$ of \mathcal{N}_F is open and has finite index. More precisely, the homomorphism*

$$V_F/D(V_F) \rightarrow \mathcal{N}_F/D(\mathcal{N}_F), \quad \varphi D(V_F) \mapsto \varphi D(\mathcal{N}_F)$$

is surjective.

Proof. We first show that $D(\mathcal{N}_F)$ is open. If $U(F)$ is discrete, so is \mathcal{N}_F and there is nothing to show. Recall that $U(F)^+$ is the subgroup of $U(F)$ generated by all the edge fixators in $U(F)$. It is trivial if and only if the action of F on D is free, so if and only if $U(F)$ is discrete, see Remark 2.8. Otherwise it is open in $U(F)$ and simple by Theorem 2.4. If $U(F)$ is non-discrete, it is easy to find two non-commuting elements in $U(F)^+$, so $U(F)^+$ is not an abelian group. Therefore $D(\mathcal{N}_F) \cap U(F)^+$ is non-trivial and normal in $U(F)^+$. Simplicity of $U(F)^+$ now implies $U(F)^+ \leq D(\mathcal{N}_F)$ and as a conclusion $D(\mathcal{N}_F)$ is open.

Obviously $D(V_F)$ is a normal subgroup of $(\mathcal{N}_F) \cap V_F$. By the third isomorphism theorem the homomorphism $V_F/D(V_F) \rightarrow V_F/(D(\mathcal{N}_F) \cap V_F)$ is surjective. The second isomorphism theorem implies

$$V_F/(D(\mathcal{N}_F) \cap V_F) \cong V_F \cdot D(\mathcal{N}_F)/D(\mathcal{N}_F).$$

Since V_F is dense and $D(\mathcal{N}_F)$ is open in \mathcal{N}_F we know $V_F \cdot D(\mathcal{N}_F) = \mathcal{N}_F$ and the result follows. □

The following corollary is immediate.

Corollary 4.12. *If d is even and F is transitive, then \mathcal{N}_F is simple.*

4.1 Normal subgroups

We want to understand what normal subgroups \mathcal{N}_F can have.

Sign of an almost automorphism. Let T_1, T_2 be finite complete subtrees of \mathcal{T} . Let $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$ be a $U(F)$ -honest almost automorphism. If $\varphi \in V_F$ then by enlarging T_1 and T_2 if necessary we assume that φ is induced by the bijection $\varphi|_{\mathcal{LT}_1}: \mathcal{LT}_1 \rightarrow \mathcal{LT}_2$. Consider an F -invariant subset $D' \subset D$. Then φ induces a bijection

$$\kappa: \{v \in \mathcal{LT}_1 \mid \text{col}(v) \in D'\} \rightarrow \{v \in \mathcal{LT}_2 \mid \text{col}(v) \in D'\}.$$

Recall that in Section 2.4 we defined the lexicographical order on the plane ordered tree \mathcal{T} . There exists a unique order-preserving bijection

$$\iota: \{v \in \mathcal{LT}_2 \mid \text{col}(v) \in D'\} \rightarrow \{v \in \mathcal{LT}_1 \mid \text{col}(v) \in D'\}.$$

We define $\varphi_{\mathcal{L}_{D'}T_1} := \iota \circ \kappa$. Denote by $\text{sgn}_{D'}(\varphi) \in \{1, -1\}$ the sign of the permutation $\varphi_{\mathcal{L}_{D'}T_1}$. Recall that, as we have seen in Section 2.4 for $D' = D$, it is only defined on honest almost automorphisms and is not constant on equivalence classes in general.

Proposition 4.13. *Let $D' \subset D$ be F -invariant.*

- a) *The sign $\text{sgn}_{D'}$ induces a well-defined homomorphism $V_F \rightarrow \{1, -1\}$ if and only if the cardinality $|D'|$ is even.*
- b) *The sign $\text{sgn}_{D'}$ induces a well-defined homomorphism $\mathcal{N}_F \rightarrow \{1, -1\}$ if and only if the following two conditions are satisfied.*
 - 1. *For every $\chi \in D$ holds $\{f|_{D'} \mid f \in F: f(\chi) = \chi\} \leq \text{Alt}(D')$.*
 - 2. *The cardinality $|D'|$ is even.*

Proof. Let T_1, T_2, T_3 be finite complete subtrees of \mathcal{T} and let

$$\begin{aligned} \psi: \mathcal{T} \setminus T_1 &\rightarrow \mathcal{T} \setminus T_2 \\ \psi': \mathcal{T} \setminus T_2 &\rightarrow \mathcal{T} \setminus T_3 \end{aligned}$$

be $U(F)$ -honest almost automorphisms. It is clear that

$$\text{sgn}_{D'}(\psi' \circ \psi) = \text{sgn}_{D'}(\psi') \text{sgn}_{D'}(\psi).$$

We now prove the “if”-parts of a) and b). Consider a $U(F)$ -honest almost automorphism $\varphi: \mathcal{T} \setminus T_1 \rightarrow \mathcal{T} \setminus T_2$. We need to show that for an equivalent honest almost automorphism φ' with T_1 replaced by a simple expansion T'_1 (and T_2 replaced by a simple expansion T'_2) holds $\text{sgn}_{D'}(\varphi) = \text{sgn}_{D'}(\varphi')$. Recall that an inversion of the permutation $\varphi_{\mathcal{L}_{D'}T_1}$ is a pair (v, w) such that $v < w$ but $\varphi_{\mathcal{L}_{D'}T_1}(v) > \varphi_{\mathcal{L}_{D'}T_1}(w)$. Also recall that the sign of a permutation is 1 or -1 depending on if the number of its inversions is even or odd. Denote the set of inversions of a permutation ρ by $\text{Inv}(\rho)$.

Let $w_0 \in \mathcal{LT}_1$ be the leaf of T_1 whose children are leaves of T'_1 .

Step 1: The “if”-part of a). Assume that $|D'|$ is even. Assume that the $U(F)$ -honest almost automorphism φ is the element of V_F induced by the bijection $\varphi|_{\mathcal{LT}_1} : \mathcal{LT}_1 \rightarrow \mathcal{LT}_2$.

Observe that

$$\begin{aligned} \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) &= \{(v, w) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid v \neq w_0 \neq w\} \\ &\sqcup \{(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) \mid v \text{ child of } w_0, w \text{ no child of } w_0\} \\ &\sqcup \{(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) \mid v \text{ no child of } w_0, w \text{ child of } w_0\} \\ &\sqcup \{(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) \mid v, w \text{ children of } w_0\}. \end{aligned}$$

The second assumption of Step 1 implies

$$\{(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) \mid v, w \text{ children of } w_0\} = \emptyset.$$

Let w be a child of w_0 and $v \in \mathcal{LT}_1 \setminus \{w_0\}$ such that $\text{col}(v), \text{col}(w) \in D'$. Then (v, w) is an inversion for $\varphi'_{\mathcal{L}_{D',T'_1}}$ if and only if for every child w' of w_0 with $\text{col}(w') \in D'$ the pair (v, w') is an inversion. The analogous statement holds for the pair (w, v) . Therefore the cardinalities of the second and third set in above union are divisible by the number of children of w_0 whose colour is in D' . We now distinguish two cases.

Case 1: $\text{col}(w_0) \notin D'$

In this case

$$\{(v, w) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid v \neq w_0 \neq w\} = \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}).$$

Furthermore the number of children of w_0 whose colour is in D' is $|D'|$, hence even by assumption. Therefore $|\text{Inv}(\varphi_{\mathcal{L}_{D',T_1}})|$ is even if and only if $|\text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}})|$ is even. Consequently $\text{sgn}_{D'}(\varphi') = \text{sgn}_{D'}(\varphi)$.

Case 2: $\text{col}(w_0) \in D'$

Let $v \neq w_0$ be a leaf of T_1 with $\text{col}(v) \in D'$. Let w be a child of w_0 with $\text{col}(w) \in D'$. Note that by definition of the lexicographical order holds $(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}})$ if and only if $(v, w_0) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}})$. This implies

$$\begin{aligned} &|\{(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) \mid v \text{ child of } w_0, w \text{ no child of } w_0\}| \\ &= (|D'| - 1) \cdot |\{(v, w) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid w = w_0\}| \end{aligned}$$

and since, by assumption, the number $|D'| - 1$ is odd, the cardinality $|\{(v, w) \in \text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}}) \mid v \text{ child of } w_0, w \text{ no child of } w_0\}|$ is even if and only if $|\{(v, w) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid w = w_0\}|$ is even. The analogous statement holds for (w, v) instead of (v, w) . Consequently

$$\begin{aligned} |\text{Inv}(\varphi'_{\mathcal{L}_{D',T'_1}})| &\equiv |\{(v, w) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid v \neq w_0 \neq w\}| \\ &\quad + |\{(v, w) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid w = w_0\}| \\ &\quad + |\{(w, v) \in \text{Inv}(\varphi_{\mathcal{L}_{D',T_1}}) \mid w = w_0\}| \\ &\equiv |\text{Inv}(\varphi_{\mathcal{L}_{D',T_1}})| \pmod{2}. \end{aligned}$$

This implies $\text{sgn}_{D'}(\varphi') = \text{sgn}_{D'}(\varphi)$.

Step 2: The “if”-part of b). Assume that Assumptions 1. and 2. hold.

Recall that the bijection $\varphi|_{\mathcal{LT}_1} : \mathcal{LT}_1 \rightarrow \mathcal{LT}_2$ uniquely determines an element of V_F . By postcomposing φ with the inverse of this element and Step 1, we can assume $\varphi|_{\mathcal{LT}_1} = \text{id}$. Now passing to φ' , we see that the only possible inversions for $\varphi'|_{\mathcal{L}_{D'}T'_1}$ are amongst the D' -coloured children of w_0 . Note that they are permuted by an element of F that fixes w_0 . By Assumption 2., there are evenly many inversions. This concludes the proof that $\text{sgn}_{D'}(\varphi) = \text{sgn}_{D'}(\varphi')$.

For the “only if”-parts denote by $S_n(v_0)$ the vertices of distance n to v_0 and denote by $B_n(v_0)$ the finite complete subtree of \mathcal{T} spanned by $S_n(v_0)$.

Step 3: The “only if”-part of a). Assume that $|D'|$ is odd.

Let $\chi \in D'$. Let $n \geq 2$ be an integer and let $v, w \in S_n(v_0)$ be such that $\text{col}(v) = \text{col}(w) = \chi$. Let $\varphi : \mathcal{T} \setminus B_n(v_0) \rightarrow \mathcal{T} \setminus B_n(v_0)$ be the element of V_F induced by the transposition of v and w . Then $\varphi|_{\mathcal{L}_{D'}B_n(v_0)}$ is a transposition and therefore $\text{sgn}_{D'}(\varphi) = -1$. We consider now the $U(F)$ -honest almost automorphism $\varphi' : \mathcal{T} \setminus B_{n+1}(v_0) \rightarrow \mathcal{T} \setminus B_{n+1}(v_0)$ that is equivalent to φ . The permutation $\varphi'|_{\mathcal{L}_{D'}B_{n+1}(v_0)}$ is the product of $|D'| - 1$ many transpositions. Since $|D'|$ is odd, $\text{sgn}_{D'}(\varphi') = 1$. Therefore $\text{sgn}_{D'}$ is not well-defined on equivalence classes of almost automorphisms.

Step 4: The “only if”-part of b). Assume that Assumption 1. does not hold.

Let $n \geq 1$. Let $f \in F$ and $\chi \in D$ be such that $f(\chi) = \chi$ and such that $f|_{D'} \notin \text{Alt}(D')$. Choose an element $g \in U(F)$ as follows. Pick $v \in S_n(v_0)$ with $\text{col}(v) = \chi$. Let $g|_{\mathcal{T} \setminus \mathcal{T}_v} = \text{id}$ and $\text{prm}_{g,v} = f$. Note that this implies $g(v) = v$ and $g(\mathcal{T}_v) = \mathcal{T}_v$. Look at $U(F)$ -honest almost automorphisms $\varphi : \mathcal{T} \setminus B_n(v_0) \rightarrow \mathcal{T} \setminus B_n(v_0)$ and $\varphi' : \mathcal{T} \setminus B_{n+1}(v_0) \rightarrow \mathcal{T} \setminus B_{n+1}(v_0)$ equivalent to g . Then, since f is an odd permutation we know that $\text{sgn}_{D'}(\varphi) = 1$ and $\text{sgn}_{D'}(\varphi') = -1$. Therefore $\text{sgn}_{D'}$ is not well-defined on the equivalence classes of almost automorphisms. \square

Corollary 4.14. *If the abelianizations of \mathcal{N}_F and V_F are isomorphic, then for every F -invariant subset D' of D with even cardinality and every $\chi \in D$ the restricted point stabilizer $\{f|_{D'} \mid f(\chi) = \chi\}$ is contained in $\text{Alt}(D')$.*

Example 4.15. Let $d = 6$ and let $F = \langle (12)(34), (56) \rangle$. Then F has four orbits, one of them with odd cardinality, and therefore $[V_F : D(V_F)] = 8$. Consider $0 \in D$. Its stabilizer in F equals F and restricts to a subgroup of $\text{Alt}(D')$ with $|D'|$ even for $D' = \{1, 2, 3, 4\}$. The stabilizer of $1, 2, 3, 4$ restricts to a subgroup of $\text{Alt}(D')$ with $|D'|$ even for $D' = \{1, 2, 3, 4\}$. The stabilizer of 5 and 6 restricts to a subgroup of $\text{Alt}(D')$ with $|D'|$ even for $D' = \{1, 2, 3, 4\}$ and $D' = \{1, 2, 3, 4, 5, 6\}$. Therefore $\text{sgn}_{D'}$ is a well-defined homomorphism only for $D' = \{1, 2, 3, 4\}$.

Remark 4.16. The set $\Delta := \{D' \subset D \mid D' \text{ as in Prop. 4.13b)}\}$ is closed under symmetric difference, so it is an abelian group where every element has order 2. It seems plausible, and is true for V_F , that $\mathcal{N}_F/D(\mathcal{N}_F)$ is isomorphic to this group via an isomorphism induced by

$$\mathcal{N}_F \rightarrow \Delta, \quad \varphi \mapsto \sum_{\text{sgn}_{D'}(\varphi)=-1} D'.$$

5 No lattices

Let $F \leq \text{Sym}(D)$ be such that $U(F)$ is non-discrete, i.e. such that F does not act freely on D . Recall that we call F a *single-switch group* if it has order 2 and its only non-trivial element is a transposition. Denote $\mathcal{N}_F := \mathcal{N}(U(F))$. The main goal of this section is to prove the following theorem.

Theorem 5.1. *Assume F is not a single switch group. Then the group \mathcal{N}_F does not admit a lattice.*

The case $F = \text{Sym}(D)$ is the content of [BCGM12]. Our proof follows the same argument.

Remark 5.2. Le Boudec proves for certain $F \leq \text{Sym}(D)$, which are transitive and satisfy additional assumptions that we do not want to elaborate on here, that $G(F) := \mathcal{N}_F \cap \text{Aut}(\mathcal{T})$ does not have any lattice, see [LB16], Corollary 7.7. Since $G(F) < \mathcal{N}_F$ is an open subgroup this implies our result by the following well-known lemma.

Lemma 5.3. *Let Λ be a locally compact group and $\tilde{\Lambda} \leq \Lambda$ be an open subgroup. Let Γ be a lattice in Λ . Then $\tilde{\Lambda} \cap \Gamma$ is a lattice in $\tilde{\Lambda}$.*

Proof. We first show that the obvious inclusion $i: \tilde{\Lambda}/\tilde{\Lambda} \cap \Gamma \hookrightarrow \Lambda/\Gamma$ is an open map. Denote by $p: \tilde{\Lambda} \rightarrow \tilde{\Lambda}/\tilde{\Lambda} \cap \Gamma$ and $q: \Lambda \rightarrow \Lambda/\Gamma$ the quotient maps. Let $U \subset \tilde{\Lambda}/\tilde{\Lambda} \cap \Gamma$ be an open set. Then $p^{-1}(U)$ is open in $\tilde{\Lambda}$ and therefore also in Λ . We see that

$$\begin{aligned} q(p^{-1}(U)) &= q(\{g \in \tilde{\Lambda} \mid g(\tilde{\Lambda} \cap \Gamma) \in U\}) \\ &= \{g\Gamma \in \tilde{\Lambda} \mid g(\tilde{\Lambda} \cap \Gamma) \in U\} \\ &= i(U). \end{aligned}$$

Also note that for every open subset $U' \subset \Lambda$ holds $q^{-1}(q(U')) = U'\Gamma$. Therefore by the definition of the quotient topology the map q is open. Consequently i is an open map. Because i is $\tilde{\Lambda}$ -equivariant we can restrict a finite Λ -invariant measure on Γ/Λ to a $\tilde{\Lambda}$ -invariant measure on $\tilde{\Gamma}/\tilde{\Gamma} \cap \Lambda$. The result follows. \square

Notation 5.4. As in the preceding section we denote the orbits of F by $D^{(0)}, \dots, D^{(l)} \subset D$. For each $i = 0, \dots, l$ we write $d^{(i)} := |D^{(i)}|$.

For $n \geq 0$ we denote by $S_n(v_0)$ the set of vertices of distance n from v_0 and by $B_n(v_0)$ the subtree spanned by $S_n(v_0)$, i.e. the smallest subtree of \mathcal{T} containing $S_n(v_0)$.

Denote by O_n the equivalence classes of all $U(F)$ -honest almost automorphisms $\mathcal{T} \setminus B_n(v_0) \rightarrow \mathcal{T} \setminus B_n(v_0)$. It is easy to see that $O_n < \mathcal{N}_F$ is a compact and open subgroup and that $O_n < O_{n+1}$. For $n = 0$ we have $B_0(v_0) = \{v_0\}$ and $O_0 = \text{Fix}_{U(F)}(v_0) < U(F)$. Let

$$O = \bigcup_{n \geq 0} O_n.$$

Denote by μ the Haar measure on \mathcal{N}_F normalized by $\mu(O_0) = 1$. For $n \geq 0$ denote

$$U_n = \text{Fix}_{U(F)}(B_n(v_0)) < U(F).$$

In particular $U_0 = O_0$. The collection $\{U_n \mid n \geq 0\}$ is a neighbourhood basis of the identity for $U(F)$ and therefore also for \mathcal{N}_F and O .

For $n \geq 1$ the group O_n acts on the $(d+1)d^{n-1}$ leaves of $B_n(v_0)$. Denote this action by

$$\pi_n: O_n \rightarrow \text{Sym}(S_n(v_0)).$$

Its kernel is U_n . With Lemma 2.9 we can see that it has $l+1$ orbits $D_n^{(0)}, \dots, D_n^{(l)}$. We can determine $D_n^{(i)}$ explicitly, namely

$$D_n^{(i)} = \{v \in S_n(v_0) \mid \text{col}(v) \in D^{(i)}\}.$$

Since O_n preserves the partition $S_n(v_0) = \bigsqcup_{i=0}^l D_n^{(i)}$, acts on each $D_n^{(i)}$ as the whole symmetric group and for $i \neq j$ permutes the vertices of $D_n^{(i)}$ and $D_n^{(j)}$ independently, the image of π_n is $\prod_{i=0}^l \text{Sym}(D_n^{(i)})$. Therefore π_n induces an isomorphism

$$O_n/U_n \cong \prod_{i=0}^l \text{Sym}(D_n^{(i)}).$$

5.1 The group O has no lattice

Since $O < \mathcal{N}_F$ is open, the following theorem directly implies Theorem 5.1 by Lemma 5.3.

Theorem 5.5. *Assume F is not a single switch group. Then the group O does not admit any lattice.*

Remark 5.6 (*Strategy of the proof of Theorem 5.5*). Let by contradiction $\Gamma < O$ be a lattice. Denote its covolume by c . Similarly denote by c_n the covolume of $\Gamma \cap O_n$ in O_n . We will now establish a lower bound for c in

terms of the index of $\Gamma_n := \pi_n(\Gamma \cap O_n)$ in $\prod_{i=0}^l \text{Sym}(D_n^{(i)})$ and use it to get an estimate for the index $[\text{Sym}(S_n(v_0)) : \Gamma_n]$. Using this estimate we will, precisely as in [BCGM12], find non-trivial elements in $\Gamma \cap U_n$ for very large n , which shows that Γ cannot be discrete.

Notation 5.7. Recall that we denoted $G(F) = \mathcal{N}_F \cap \text{Aut}(\mathcal{T})$. More explicitly

$$G(F) = \{g \in \text{Aut}(\mathcal{T}) \mid |\{v \in \mathcal{V} \mid \text{prm}_{g,v} \notin F\}| < \infty\}.$$

For a subgroup $G \leq \text{Aut}(\mathcal{T})$ we denote

$$\text{Aut}_G(B_n(v_0)) = \{g \in \text{Aut}(B_n(v_0)) \mid \exists h \in G: g = h|_{B_n(v_0)}\}.$$

Covolume estimate. Since Γ is discrete there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ holds $\Gamma \cap U_n = \{1\}$. That implies $\Gamma \cap O_n \cong \pi_n(\Gamma \cap O_n) =: \Gamma_n$ since $U_n = \ker(\pi_n)$. We can make for $n \geq n_0$ the volume computation

$$\begin{aligned} c \geq c_n = \text{vol}(O_n / \Gamma \cap O_n) &= \frac{\mu(O_n)}{|\Gamma \cap O_n|} = \frac{[O_n : O_0]}{|\Gamma_n|} = \frac{[O_n : U_n]}{|\Gamma_n| \cdot [U_0 : U_n]} \\ &= \frac{|\prod_{i=0}^l \text{Sym}(D_n^{(i)})|}{|\Gamma_n| \cdot |\text{Aut}_{U(F)}(B_n(v_0))|} \geq \frac{|\prod_{i=0}^l \text{Sym}(D_n^{(i)})|}{|\Gamma_n| \cdot |\text{Aut}_{G(F)}(B_n(v_0))|} \\ &= \frac{[\prod_{i=0}^l \text{Sym}(D_n^{(i)}) : \Gamma_n]}{|\text{Aut}_{G(F)}(B_n(v_0))|}. \end{aligned} \tag{1}$$

To prove that Γ cannot exist, we need a preparatory estimate.

Proposition 5.8. *If $l < d - 1$ and $d > 2$ there exists a constant $C = C(d)$ such that for n big enough holds*

$$[\text{Sym}(S_n(v_0)) : \pi_n(\Gamma)] \leq C \cdot d^{|S_n(v_0)|}.$$

Remark 5.9. One can check that if $l = d - 1$ then the inequality is reversed. This corresponds to exactly l of the numbers $d^{(i)}$ being equal to 1 and the remaining one equal to 2. This is precisely the case where F is a single switch group.

The case $F = \text{Sym}(D)$, in particular the case $d = 2$ and $l = 0$, is covered in [BCGM12].

Before going to the quite technical proof of this Proposition we will derive Theorem 5.5 from it. We rephrase the key proposition from [BCGM12], which roughly says that subgroups of a huge finite symmetric group satisfying a certain index bound must contain one large alternating group or a product of many not so small alternating groups.

Proposition 5.10 (*Proposition 4.1 from [BCGM12]*). *Let $c, d > 0$ be positive real numbers and $0 < \alpha < 1$. There exists an integer n_1 depending on c, d and α such that for every finite set K with $|K| \geq n_1$ every subgroup $\Lambda \leq \text{Sym}(K)$ with*

$$[\text{Sym}(K) : \Lambda] \leq c \cdot d^{|K|}$$

satisfies one of the following (non-exclusive) alternatives.

1. *There exists a subset $Z \subset K$ with $|Z| > \frac{|K|}{d} + 2$ and $\text{Alt}(Z) \leq \Lambda$.*
2. *There exist d disjoint subsets $Z_1, \dots, Z_d \subset K$ which satisfy*

$$\left| \bigsqcup_{j=1}^d Z_j \right| > (1 - \alpha)|K| \quad \text{and} \quad \prod_{j=1}^d \text{Alt}(Z_j) \leq \Lambda.$$

The conclusion of the proof that O does not have a lattice works exactly as in [BCGM12]. For completeness we reproduce it here.

Proof of Theorem 5.5. Let $\alpha < 1/d^2$. By Proposition 5.8 we may apply Proposition 5.10 to $K = S_n(v_0)$ and $\Lambda = \Gamma_n$ with C, d and α for some fixed $n \geq \max\{n_0 + 2, n_1\}$. Note that the choice of n implies $\Gamma \cap U_{n-2} = \{1\}$.

We introduce some terminology. Vertices with same parent are called siblings. Grandparents and grandchildren are defined in the obvious way.

First assume that Γ_n satisfies Alternative 1. Then, by the pigeonhole principle there need to exist either three siblings v_1, v_2, v_3 or two pairs of siblings w_1, w_2 and w_3, w_4 in the set Z . Since $\text{Alt}(Z) \leq \Gamma_n$, the corresponding permutation $(v_1 v_2 v_3) \in \text{Alt}(Z)$ or $(w_1 w_2)(w_3 w_4) \in \text{Alt}(Z)$ is in Γ_n . These permutations only permute amongst siblings, so the preimage under π_n of this element must be a nontrivial element in $U_{n-1} \cap \Gamma$. This contradicts $n - 1 \geq n_0$.

Assume now that Γ_n satisfies Alternative 2. We can assume in addition that Γ_n does not contain a nontrivial element that only permutes siblings, because otherwise we get a contradiction as above. This means that every Z_j contains at most one pair of siblings. We call siblings that are contained in the same Z_j twins. Note that there are as many Z_j as every parent has children. So, if a parent does not have twins, but still all its children are contained $Z := \bigsqcup_{j=1}^d Z_j$, then every Z_j contains exactly one of their children. Note that there are at most d parents of twins and thus also at most d grandparents of twins.

There are at most $\alpha \cdot |S_n(v_0)|$ vertices in $S_{n-1}(v_0)$ having a grandchild that is not in Z . Since $\alpha < 1/d^2$ and $|S_n(v_0)|$, this means that at least $(1/d^2 - \alpha) \cdot |S_n(v_0)|$ grandparents in $S_{n-1}(v_0)$ have all their grandchildren in Z . If n is such that $(1/d^2 - \alpha) \cdot |S_n(v_0)| \geq d + 2$, there are at least two grandparents $g_1, g_2 \in S_{n-2}(v_0)$ all of whose grandchildren are in Z but who are not grandparents of twins. For each of the two g_i , we can construct an

element in $\prod_{j=1}^d \text{Sym}(Z_j)$ by switching two of their children in a way that the grandchildren do not change Z_j they are contained in. By composing these two elements, we get an element in $\prod_{j=1}^l \text{Alt}(Z_j) \leq \Gamma_n$ whose preimage under π_n lies in $U_{n-2} \cap \Gamma$, contradiction. \square

In the proof of Proposition 5.8 we will need the following formulae.

Lemma 5.11. *For $n \geq 1$ holds*

$$\left| \prod_{i=0}^l \text{Sym}(D_n^{(i)}) \right| = \prod_{i=0}^l (d^{(i)} \cdot d^{n-1})!$$

$$|\text{Aut}_{G(F)}(B_n(v_0))| = \left(\prod_{i=0}^l d^{(i)}! \right) \cdot \left(\prod_{i=0}^l \frac{d^{(i)}!(d+1)}{d^{(i)d^{(i)}}} \right)^{\frac{d^{n-1}-1}{d-1}}.$$

Proof. By symmetry, for every $\chi \in D$ the number of vertices $v \in S_n(v_0)$ with $\text{col}(v) = \chi$ is $\frac{|S_n(v_0)|}{|D|} = d^{n-1}$. This implies $d_n^{(i)} = d^{(i)} \cdot d^{n-1}$ and

$$\left| \prod_{i=0}^l \text{Sym}(D_n^{(i)}) \right| = \prod_{i=0}^l (d^{(i)} \cdot d^{n-1})!.$$

The group $\text{Aut}_{G(F)}(B_n(v_0))$ consists of all those automorphisms $g \in B_n(v_0)$ such that for every $v \in \mathcal{V}$ the local permutation $\text{prm}_{g,v}$ preserves the orbits $D^{(0)}, \dots, D^{(l)}$, see Lemma 2.9 and Lemma 2.19. Restriction to $B_{n-1}(v_0)$ defines a surjective homomorphism

$$\rho_n: \text{Aut}_{G(F)}(B_n(v_0)) \rightarrow \text{Aut}_{G(F)}(B_{n-1}(v_0)).$$

The kernel of ρ_n consists of all those automorphisms of $B_n(v_0)$ which fix $B_{n-1}(v_0)$.

Incident to each leaf of colour in $D^{(i)}$ in $B_{n-1}(v_0)$ are $d^{(j)}$ leaves of $B_n(v_0)$ with colour in $D^{(j)}$ for $j \neq i$ and $d^{(i)} - 1$ leaves with colour in $D^{(i)}$. Thus we get $|\text{Aut}_{G(F)}(B_1(v_0))| = \prod_{i=0}^l d^{(i)}!$ and for $n \geq 2$ we see

$$\begin{aligned} |\text{Ker}(\rho_n)| &= \prod_{i=0}^l \prod_{D_{n-1}^{(i)}} (d^{(i)} - 1)! \prod_{j \neq i} d^{(j)}! \\ &= \prod_{i=0}^l \left(\frac{\prod_{j=0}^l d^{(j)}!}{d^{(i)}} \right)^{d_{n-1}^{(i)}} \\ &= \left(\prod_{i=0}^l \frac{d^{(i)}!(d+1)}{d^{(i)d^{(i)}}} \right)^{d^{n-2}}. \end{aligned}$$

Inductively we get

$$\begin{aligned}
|\operatorname{Aut}_{G(F)}(B_n(v_0))| &= |\operatorname{Aut}_{G(F)}(B_{n-1}(v_0))| \cdot |\operatorname{Ker}(\rho_k)| \\
&= |\operatorname{Aut}_{G(F)}(B_1(v_0))| \cdot \prod_{k=2}^n |\operatorname{Ker}(\rho_n)| \\
&= |\operatorname{Aut}_{G(F)}(B_1(v_0))| \cdot \prod_{k=2}^n \left(\frac{\prod_{j=0}^l d^{(j)}!^{(d+1)}}{\prod_{i=0}^l d^{(i)d^{(i)}}} \right)^{d^{k-2}} \\
&= \left(\prod_{i=0}^l d^{(i)}! \right) \cdot \left(\prod_{i=0}^l \frac{d^{(i)}!^{(d+1)}}{d^{(i)d^{(i)}}} \right)^{\frac{d^{n-1}-1}{d-1}}.
\end{aligned}$$

□

Proof of Proposition 5.8. Recall Estimate (1), namely

$$\left[\prod_{i=0}^l \operatorname{Sym}(D_n^{(i)}) : \pi_n(\Gamma) \right] \leq c \cdot |\operatorname{Aut}_{G(F)}(B_n(v_0))|.$$

It is equivalent to

$$[\operatorname{Sym}(S_n(v_0)) : \pi_n(\Gamma)] \leq c \cdot |\operatorname{Aut}_{G(F)}(B_n(v_0))| \cdot \left[\operatorname{Sym}(S_n(v_0)) : \prod_{i=0}^l \operatorname{Sym}(D_i^n) \right],$$

so it suffices to show that

$$|\operatorname{Aut}_{G(F)}(B_n(v_0))| \cdot \left[\operatorname{Sym}(S_n(v_0)) : \prod_{i=0}^l \operatorname{Sym}(D_i^n) \right] \leq C \cdot d^{|B_n(v_0)|}.$$

We write out this inequality explicitly in the values calculated in Lemma 5.11 and use the O -Notation. The term $\prod_{i=0}^l d^{(i)}!$ is constant, so the inequality is equivalent to

$$\left(\prod_{i=0}^l \frac{d^{(i)}!^{(d+1)}}{d^{(i)d^{(i)}}} \right)^{\frac{d^{n-1}-1}{d-1}} \cdot \frac{((d+1)d^{n-1})!}{\prod_{i=0}^l d_n^{(i)}!} \leq O(1) \cdot d^{(d+1)d^{n-1}}$$

and after taking the logarithm it is equivalent to

$$\begin{aligned}
&\frac{d^{n-1}-1}{d-1} \sum_{i=0}^l \left((d+1) \ln(d^{(i)}!) - d^{(i)} \ln(d^{(i)}) \right) + \ln(((d+1)d^{n-1})!) \\
&\leq (d+1)d^{n-1} \ln(d) + \sum_{i=0}^l \ln(d_n^{(i)}!) + O(1).
\end{aligned}$$

We need to deal with factorials of powers. For that we invoke Stirling's estimate

$$\sqrt{2\pi} \cdot m^{m+\frac{1}{2}} e^{-m} \leq m! \leq \sqrt{e} \cdot m^{m+\frac{1}{2}} e^{-m}$$

showing that

$$\ln(m!) = \left(m + \frac{1}{2}\right) \ln(m) - m + O(1).$$

It yields

$$\begin{aligned} \ln(((d+1)d^{n-1})!) &= \left((d+1)d^{n-1} + \frac{1}{2}\right) \ln((d+1)d^{n-1}) \\ &\quad - (d+1)d^{n-1} + O(1) \\ &= n \cdot d^{n-1} ((d+1) \ln(d)) \\ &\quad + d^{n-1}(d+1)(\ln(d+1) - \ln(d) - 1) + O(n) \end{aligned}$$

and

$$\begin{aligned} \ln(d_n^{(i)}!) &= \ln\left((d^{(i)}d^{n-1})!\right) \\ &= \left(d^{(i)}d^{n-1} + \frac{1}{2}\right) \ln(d^{(i)}d^{n-1}) - d^{(i)}d^{n-1} + O(1) \\ &= n \cdot d^{n-1} \left(d^{(i)} \ln(d)\right) \\ &\quad + d^{n-1}d^{(i)} \left(\ln(d^{(i)}) - \ln(d) - 1\right) + O(n). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\frac{d^{n-1}}{d-1} \sum_{i=0}^l \left((d+1) \ln(d^{(i)}!) - d^{(i)} \ln(d^{(i)}) \right) \\ &\quad + n \cdot d^{n-1} ((d+1) \ln(d)) \\ &\quad + d^{n-1}(d+1)(\ln(d+1) - \ln(d) - 1) + O(n) \\ &\leq (d+1)d^{n-1} \ln(d) \\ &\quad + \sum_{i=0}^l \left(n \cdot d^{n-1} \left(d^{(i)} \ln(d)\right) + d^{n-1}d^{(i)} \left(\ln(d^{(i)}) - \ln(d) - 1\right) \right). \end{aligned}$$

The dominant term nd^{n-1} appears on both sides with the same coefficient $(d+1) \ln(d)$, so we can eliminate it and compare the coefficients of the newly dominating term, namely d^{n-1} . They are

$$\frac{1}{d-1} \sum_{i=0}^l \left((d+1) \ln(d^{(i)}!) - d^{(i)} \ln(d^{(i)}) \right) + (d+1)(\ln(d+1) - \ln(d) - 1)$$

on the left and

$$(d+1) \ln(d) + \sum_{i=0}^l \left(d^{(i)} \left(\ln(d^{(i)}) - \ln(d) - 1 \right) \right) = \sum_{i=0}^l \left(d^{(i)} \ln(d^{(i)}) \right) - (d+1)$$

on the right. To conclude the proof it suffices to show that this left dominant coefficient is strictly smaller than this right dominant coefficient, which is equivalent to

$$\frac{d+1}{d-1} \sum_{i=0}^l \left(\ln(d^{(i)!}) \right) + (d+1)(\ln(d+1) - \ln(d)) < \frac{d}{d-1} \sum_{i=0}^l \left(d^{(i)} \ln(d^{(i)}) \right),$$

i.e.

$$(d-1)(\ln(d+1) - \ln(d)) < \frac{d}{d+1} \sum_{i=0}^l d^{(i)} \ln(d^{(i)}) - \sum_{i=0}^l \ln(d^{(i)!}).$$

This inequality is precisely the content of the next lemma. \square

Lemma 5.12. *If $l < d-1$ and $d > 2$, then*

$$(d-1)(\ln(d+1) - \ln(d)) < \frac{d}{d+1} \sum_{i=0}^l d^{(i)} \ln(d^{(i)}) - \sum_{i=0}^l \ln(d^{(i)!}).$$

Proof. For $d = 2$ and $l = 0$ we have equality. If $l = 1$ and $(d^{(0)}, d^{(1)}) = (2, 2)$ the inequality is true. Inductively and by symmetry in $d^{(0)}, \dots, d^{(k)}$, the lemma will follow from two claims.

For positive integers $x^{(0)}, \dots, x^{(k)}$ with $\sum_{i=0}^k x^{(i)} = x+1$ and $k < x$ define the function Ξ as the difference of the right hand side minus the left hand side, i.e.

$$\Xi(x^{(0)}, \dots, x^{(k)}) = \frac{x}{x+1} \sum_{i=0}^k x^{(i)} \ln(x^{(i)}) - \sum_{i=0}^k \ln(x^{(i)!}) + (x-1) \ln\left(\frac{x}{x+1}\right).$$

Claim 1: $\Xi(x^{(0)}, \dots, x^{(k)}, 1) > \Xi(x^{(0)}, \dots, x^{(k)})$

Proof: Appending 1 to the vector $(x^{(0)}, \dots, x^{(k)})$ does not change the sums $\sum_{i=0}^k x^{(i)} \ln(x^{(i)})$ and $\sum_{i=0}^k \ln(x^{(i)!})$, but increments x by 1. After obvious simplifications and rearrangings of terms the desired inequality $\Xi(x^{(0)}, \dots, x^{(k)}, 1) > \Xi(x^{(0)}, \dots, x^{(k)})$ is equivalent to

$$\left(\frac{x+1}{x+2} - \frac{x}{x+1} \right) \sum_{i=0}^k x^{(i)} \ln(x^{(i)}) > x \ln\left(\frac{x+2}{x+1}\right) - (x-1) \ln\left(\frac{x+1}{x}\right).$$

Because the function $(x^{(0)}, \dots, x^{(k)}) \mapsto \sum_{i=0}^k x^{(i)} \ln(x^{(i)})$ is convex and symmetric in $x^{(0)}, \dots, x^{(k)}$, it attains its minimum if all the $x^{(i)}$ are the same, i.e.

$$\sum_{i=0}^k x^{(i)} \ln(x^{(i)}) \geq (k+1) \frac{x+1}{k+1} \ln\left(\frac{x+1}{k+1}\right) \geq (x+1) \ln\left(\frac{x+1}{x-1}\right).$$

We estimate the term dependent on the $x^{(i)}$ from below

$$\begin{aligned} \left(\frac{x+1}{x+2} - \frac{x}{x+1} \right) \sum_{i=0}^k x_i \ln(x_i) &\geq \left(\frac{x+1}{x+2} - \frac{x}{x+1} \right) (x+1) \ln \left(\frac{x+1}{x-1} \right) \\ &= \frac{1}{x+2} \ln \left(\frac{x+1}{x-1} \right) \end{aligned}$$

and are left with showing that

$$\xi(x) := \frac{1}{x+2} \ln \left(\frac{x+1}{x-1} \right) - x \ln \left(\frac{x+2}{x+1} \right) + (x-1) \ln \left(\frac{x+1}{x} \right) > 0.$$

Observe that

$$\lim_{x \rightarrow \infty} x \ln \left(\frac{x+2}{x+1} \right) = \lim_{x \rightarrow \infty} \ln \left(\frac{\left(1 + \frac{1}{x+1}\right)^{x+1}}{1 + \frac{1}{x+1}} \right) = 1$$

and similarly

$$\lim_{x \rightarrow \infty} (x-1) \ln \left(\frac{x+1}{x} \right) = 1.$$

Therefore $\xi(x)$ converges to 0 as x approaches infinity. The first three derivatives of ξ are

$$\begin{aligned} \xi'(x) &= -\frac{\ln \left(\frac{x+1}{x-1} \right)}{(x+2)^2} - \ln \left(\frac{x}{x+1} \right) - \frac{x^2 - x + 2}{x(x+2)(x^2-1)} - \ln \left(\frac{x+2}{x+1} \right) \\ \xi''(x) &= -\frac{x^4 - 10x^3 - 15x^2 + 8x + 4}{x^2(x+2)^2(x^2-1)^2} + 2 \cdot \frac{\ln \left(\frac{x+1}{x-1} \right)}{(x+2)^3} \\ \xi'''(x) &= -2 \cdot \frac{23x^5 + 25x^4 - 25x^3 - 9x^2 + 6x + 4}{x^3(x+2)^2(x^2-1)^3} - 6 \cdot \frac{\ln \left(\frac{x+1}{x-1} \right)}{(x+2)^4}. \end{aligned}$$

Since ξ''' is strictly negative for $x \geq 2$, we know that ξ' is strictly concave. In addition ξ' converges to zero, so it must be negative. This implies that ξ is a strictly decreasing function converging to zero. Therefore ξ must be positive and Claim 1 follows.

Claim 2: $\Xi(x^{(0)}, \dots, x^{(k-1)}, x^{(k)} + 1) > \Xi(x^{(0)}, \dots, x^{(k)}, 1)$

Proof: The inequality $\Xi(x^{(0)}, \dots, x^{(k-1)}, x^{(k)} + 1) > \Xi(x^{(0)}, \dots, x^{(k)}, 1)$ is, after obvious simplifications, equivalent to

$$\frac{x}{x+1} (x^{(k)} + 1) \ln(x^{(k)} + 1) - \ln((x^{(k)} + 1)!) > \frac{x}{x+1} x^{(k)} \ln(x^{(k)}) - \ln(x^{(k)}!),$$

which after exponentiating is equivalent to

$$\left(\frac{(x^{(k)} + 1)^{x^{(k)} + 1}}{x^{(k) x^{(k)}}} \right)^{\frac{x}{x+1}} > x^{(k)} + 1.$$

We estimate the left hand side from above by setting $x = x^{(k)}$ and get

$$\left(1 + \frac{1}{x}\right)^{x^2} > x + 1$$

which is true for $x \geq 2$ because $(1 + \frac{1}{x})^x > 2$ and $2^x > x + 1$, so Claim 2 follows.

To conclude the lemma from these two claims, observe that any vector (d_0, \dots, d_l) as in the lemma arises from (3) or (2, 2) by a sequence of operations as in Claim 1 and Claim 2 and rearranging coordinates. \square

5.2 The group $\mathcal{N}_{\langle(01)\rangle}$ has no cocompact lattice

We do not know if for a single switch group F the group \mathcal{N}_F has a non-cocompact lattice or not, but at least we can prove the non-existence of cocompact lattices. After conjugating with an element of $\text{Aut}(\mathcal{T})$ we can reduce ourselves to the case where $F = \langle(01)\rangle$.

Theorem 5.13. *The group $\mathcal{N}_{\langle(01)\rangle}$ has no cocompact lattice.*

The proof works again as in [BCGM12]. We need three lemmata.

Lemma 5.14 (Ramanujan, [Ram19]). *For every $m \geq 17$ there exist three different prime numbers in the interval $(\frac{m}{2}, m]$.*

Definition 5.15. Let K be a finite set. A subgroup of $\text{Sym}(K)$ is called *primitive* if the only partitions of K it preserves are the trivial partition $\{K\}$ and the atomic partition $\{\{k\} \mid k \in K\}$.

Lemma 5.16 ([BCGM12], Lemma 3.1.). *A subgroup of the symmetric group $\text{Sym}(K)$ generated by two prime cycles whose respective supports intersect nontrivially, but are not contained in one another, acts doubly transitively (in particular, primitively) on its support.*

Lemma 5.17 (Jordan's Theorem, see [Wie64], Theorem 13.9.). *A primitive subgroup of $\text{Sym}(K)$ containing a p -cycle for a prime number $p \leq |K| - 3$ is equal to $\text{Alt}(K)$ or $\text{Sym}(K)$.*

Proof of Theorem 5.13. We again show that already O does not have a cocompact lattice. Let, by contradiction, $\Gamma < O$ be a cocompact lattice. Since $U(F) = G(F)$, the second inequality in Estimate (1) is actually an equality. Consider now a compact fundamental domain of Γ . Because O is the increasing union of the O_n , for n big enough, the fundamental domain is contained in O_n . Thus, for n big enough, the sequence (c_n) becomes constant and all inequalities in Estimate (1) are actually equalities. This shows

that c is rational and for big n holds

$$\begin{aligned} c &= \frac{|\prod_{i=0}^l \text{Sym}(D_n^{(i)})|}{|\text{Aut}_{U(F)}(B_n(v_0))| \cdot |\Gamma_n|} \\ &= \frac{(2 \cdot d^{n-1})!(d^{n-1}!)^{d-1}}{2^{d^{n-1}} \cdot |\Gamma_n|}. \end{aligned}$$

Observe that $(2 \cdot d^{n-1})!(d^{n-1}!)^{d-1}$ has arbitrarily big odd prime factors. All all of them need to be cancelled out in the fraction above by $|\Gamma_n|$. By Lemma 5.14 there exist three different prime numbers in the closed interval $[d^{n-1} + 1, 2 \cdot d^{n-1}]$. Hence there exist primes p, q such that $p + 3 < q$. None of their squares divides $(2 \cdot d^{n-1})!(d^{n-1}!)^{d-1}$. Consequently $|\Gamma_n|$ needs to be divisible by p and by q , so by Cauchy's Theorem $\Gamma_n \leq \text{Sym}(S_n(v_0))$ contains a p -cycle and a q -cycle. Without loss of generality we can assume $\{0, 1\} = D^{(0)}$. Since $p, q > d_n^{(i)}$ for every $i \geq 1$, the mentioned p -cycle and q -cycle must be contained in $\text{Sym}(D_n^{(0)})$ and intersect nontrivially. Now conjugating the p -cycle with the q -cycle we can produce another p -cycle whose support intersects the support of the original p -cycle non-trivially and such that the union of their supports has cardinality at least $p + 3$. By Lemma 5.16 and Lemma 5.17 we deduce that Γ_n contains the alternating group of some set of vertices $V_n^{(0)} \subset D_n^{(0)}$ of size $k > d_n^{(0)}/2 + 2$.

Now as in the proof of Theorem 5.5, by the pidgeonhole principle $V_n^{(0)}$ contains two pairs of siblings (v_1, w_1) and (v_2, w_2) such that $v_i \neq w_i$ for $i = 1, 2$ and $|\{v_1, v_2, w_1, w_3\}| \geq 3$. The permutation

$$\bar{\gamma} = (v_1, w_1)(v_2, w_2)$$

is an element of $\text{Alt}(V_n^{(0)}) \subset \Gamma_n$, but its pre-image $\gamma \in \Gamma \cap O_n$ is a non-trivial element of U_{n-1} . This is not possible for large n and makes the existence of Γ impossible. \square

A More groups without lattices

Let $\mathcal{T}_{d,k} = (\mathcal{V}_{d,k}, \mathcal{E}_{d,k})$ be a rooted tree such that the root has valency k and all other vertices have valency $d + 1$. Let $D := \{1, \dots, d\}$ and call it the set of colours. Note that $|D| = d$, in contrast to $d + 1$ as in the rest of this paper. Let $F \leq \text{Sym}(D)$ be a subgroup. We now rather informally define a group $\text{AAut}_F(\mathcal{T}_{d,k})$ whose construction is similar to \mathcal{N}_F . We give an outline of how to prove that it does not have any lattice. A good introduction to $\text{AAut}_F(\mathcal{T}_{d,k})$ can be found in [LB14], Section 4.

Denote the root by v_0 . Choose a map

$$\text{col}: \{e \in \mathcal{E}_{d,k} \mid v_0 \text{ is not incident to } e\} \rightarrow D$$

such that

- col is constant on geometric edges and
- for every vertex $v \neq v_0$ of $\mathcal{T}_{d,k}$ the map col restricted to the set of child edges of v is a bijection.

Recall that in contrast to this, in the rest of the paper, the restriction of col to all edges around v_0 was a bijection. Similarly to before, we can define the groups $\text{Aut}_F(\mathcal{T}_{d,k})$ of automorphisms and $\text{AAut}_F(\mathcal{T}_{d,k})$ of almost automorphisms of $\mathcal{T}_{d,k}$ which at every vertex respects the group F . More precisely, let T_1, T_2 be two finite complete subtrees of $\mathcal{T}_{d,k}$ containing v_0 , i.e. for $i = 1, 2$ the root v_0 is a vertex, but not a leaf, of T_i , and every vertex of T_i which is not v_0 and not a leaf has valency $d + 1$. Assume in addition $|\mathcal{L}T_1| = |\mathcal{L}T_2|$. Let $\varphi: \mathcal{T}_{d,k} \setminus T_1 \rightarrow \mathcal{T}_{d,k} \setminus T_2$ be an almost automorphism. As in Section 2.3 we can form the equivalence class of every almost automorphism and get a group. For every vertex v of $\mathcal{T}_{d,k} \setminus T_1$ the almost automorphism φ restricts to a bijection from the set of child leaves of v to the set of child leaves of $\varphi(v)$ and thus induces a permutation $\text{prm}_{\varphi,v} \in \text{Sym}(D)$. We can now define

$$\begin{aligned} \text{Aut}_F(\mathcal{T}_{d,k}) &= \{g: \mathcal{T}_{d,k} \rightarrow \mathcal{T}_{d,k} \mid g(v_0) = v_0, \forall v \in \mathcal{V}_{d,k} \setminus \{v_0\}: \text{prm}_{g,v} \in F\} \\ \text{AAut}_F(\mathcal{T}_{d,k}) &= \{\varphi: \mathcal{T}_{d,k} \setminus T_1 \rightarrow \mathcal{T}_{d,k} \setminus T_2 \mid \forall \text{ vertex } v: \text{prm}_{\varphi,v} \in F\}. \end{aligned}$$

The group $\text{AAut}_F(\mathcal{T}_{d,k})$ was considered by Caprace and De Medts in [CDM11]. Le Boudec [LB14] showed the existence of a group topology on $\text{AAut}_F(\mathcal{T}_{d,k})$ such that the inclusion $\text{Aut}_F(\mathcal{T}_{d,k}) \hookrightarrow \text{AAut}_F(\mathcal{T}_{d,k})$ is continuous and open. He proved that with this topology $\text{AAut}_F(\mathcal{T}_{d,k})$ is compactly presented. It is a discrete group if and only if F is trivial.

Theorem A.1. *Let $F \leq \text{Sym}(D)$ be a subgroup. If $F \neq \{id\}$, then the group $\text{AAut}_F(\mathcal{T}_{d,k})$ does not have any lattice.*

Proof (Sketch). The proof works with precisely the same method as for Theorem 5.1 above or as for Neretin's group in [BCGM12]. Definitions carry over in an obvious way, namely

$$\begin{aligned} O_n &= \{\varphi \in \text{AAut}_F(\mathcal{T}_{d,k}) \mid \varphi: \mathcal{T}_{d,k} \setminus B_n(v_0) \rightarrow \mathcal{T}_{d,k} \setminus B_n(v_0)\} \\ O &= \bigcup_{n \geq 0} O_n < \text{AAut}_F(\mathcal{T}_{d,k}) \\ U_n &= \{g \in \text{Aut}_F(\mathcal{T}_{d,k}) \mid g|_{B_n(v_0)} = id\} \\ \mu(O_0) &:= 1 \\ \pi_n: O_n &\rightarrow \text{Sym}(S_n(v_0)) \cong O_n/U_n, \end{aligned}$$

where the latter isomorphism holds because prm is only looking at the action on children, so we can extend every bijection $S_n(v_0) \rightarrow S_n(v_0)$ to an almost

automorphism $\varphi: \mathcal{T}_{d,k} \setminus B_n(v_0) \rightarrow \mathcal{T}_{d,k} \setminus B_n(v_0)$ simply by imposing, for example, $\text{prm}_{\varphi,v} = id$ for every vertex v of $\mathcal{T}_{d,k} \setminus B_n(v_0)$.

The subgroup $O \leq \text{AAut}_F(\mathcal{T}_{d,k})$ is open and we can apply Lemma 5.3. Let by contradiction $\Gamma < O$ and let $c > 0$ be its covolume. We denote $\Gamma_n := \pi_n(O_n \cap \Gamma)$. Similar to Estimate (1) we get

$$c \geq c_n := \text{vol}(O_n/\Gamma \cap O_n) = \frac{[O_n : U_n]}{|\Gamma_n| \cdot [U_0 : U_n]} \geq \frac{[\text{Sym}(S_n(v_0)) : \Gamma_n]}{\text{Aut}(B_n(v_0))}.$$

Note that

$$\begin{aligned} |S_n(v_0)| &= k \cdot d^{n-1} \\ |\text{Aut}(B_n(v_0))| &= k! \cdot d!^{k \cdot \frac{d^n-1}{d-1}}. \end{aligned}$$

Now the inequality

$$k! \cdot d!^{k \cdot \frac{d^n-1}{d-1}} \leq k! \cdot d^{k \cdot d^{n-1}}$$

implies that there exists a constant C with

$$[\text{Sym}(S_n(v_0)) : \Gamma_n] \leq C \cdot d^{|S_n(v_0)|}.$$

Therefore we can apply Proposition 5.10 and repeat the proof of Theorem 5.5 verbatim. \square

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